SPREAD TERM STRUCTURE AND DEFAULT CORRELATION

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Abstract

The aim of this paper is to analyse default correlation and its implications for the term structures of corporate bonds and credit derivatives, extending in particular the results of Jarrow, Yu (2001) and the related literature. We first consider different characterisations of spread term structures, when the available information corresponds to the default histories of the firms. The approach is then extended to factor models, both in a static and in a dynamic framework. We discuss in details the links between default correlation and jumps in short term spreads, and how these phenomenons depend on the available information.

Keywords: Corporate Bonds, Credit Risk, Default Correlation, Jumps in Intensities, Copula, Credit Derivatives

JEL Numbers: C41, G12, G21

Structure par terme et corrélation de défaut

Résumé

Le but de cet article est d'analyser la corrélation de défaut et ses implications pour les structures par terme des obligations d'entreprises et des dérivés de crédit, en étendant en particulier les résultats obtenus par Jarrow, Yu (2001). Nous commençons par donner diverses caractérisations des structures par terme de différentiels de taux, lorsque l'information disponible correspond aux historiques de défaillance. L'approche est ensuite étendue à des modèles de défaillances à facteurs, ceux-ci pouvant ou non varier dans le temps. Nous discutons particulièrment les liens entre la corrélation de défaut et les sauts dans l'évolution des différentiels de taux à court terme, et comment ces phénomènes dependent de l'information disponible.

Mots clefs: Obligations d'entreprises, risque de crédit, corrélation de défaut, saut des intensités, copule, dérivés de crédit

Classification JEL: C41, G12, G21

1 Introduction

Traditionally the analysis of credit risk has focused on the valuation of defaultable bonds and, more recently, of more complex and more exotic credit derivatives. These applications require the specification of the joint distribution of corporate times to default. In order to illustrate the main features involved in these problems, let us first consider a portfolio of corporate bonds. This portfolio includes fixed income bonds corresponding to N firms $i = 1, ..., N$. For firm i the contractual pattern of payoffs at date t is:

$$F_{i,t+h}, \quad h = 1, ..., H,$$

where $F_{i,t+h}$ has to be paid at $t+h$ by firm i. However these payments are not certain since firm i can default before. If we assume for instance a zero-recovery rate the cash-flows which will be actually received are:

$$F_{i,t+h}\mathbb{I}_{Y_i>t+h},$$

where Y_i denotes the time to default for firm i and $\mathbb{I}$ the indicator function. Therefore they are stochastic at date t. A value at t of the whole portfolio can be derived by discounting and by predicting default. More precisely a value of the credit portfolio at date t is:

$$W_t = \sum_{i=1}^{N}\sum_{h=1}^{H} F_{i,t+h} B\left(t,h\right) P_t\left(Y_i > t+h\right), \tag{1}$$

where $B\left(t,h\right)$ denotes the price at t of the riskfree zero-coupon bond with residual maturity h, and P_t a risk neutral distribution conditionally to the information available at date t.

The value at a future date $t+k < t+1$, say[1], is:

$$W_{t+k} = \sum_{i\in J_{t+k}}\sum_{h=1}^{H} F_{i,t+h} B\left(t+k,h-k\right) P_{t+k}\left(Y_i > t+h\right),$$

where J_{t+k} denotes the set of firms which are still alive at $t+k$. The portfolio value is modified for three reasons:

i) the riskfree term structure varies in time;

ii) the structure of the population of firms included in the portfolio can change with observed default;

iii) the information is modified.

[1] $t+k$ is assumed smaller than $t+1$ to avoid the possibility of intermediate payment between t and $t+k$, and the choice of a strategy to reinvest this payment.

Clearly the determination of the current portfolio value, or of the distribution of a future portfolio value, requires some knowledge about the distribution of times to default of the different firms. This knowledge concerns not only the marginal distribution of times to default, but also their dependence, called default correlation in the literature [see e.g. Duffie, Singleton (1998), Li (2000), Gouriéroux, Monfort (2002)]. Indeed defaults can arise by cluster, which induces very special dynamics of the remaining population J_{t+k}. Moreover the default of some firms can modify our beliefs on default of the other firms [an effect of the information set on the conditional probability $P_{t+k}(Y_i > t+h)$].

The need for a joint analysis of default is even increased when the portfolio includes also credit derivatives written on several default times, such as first-to-default baskets. In particular, default correlation has an effect on the term structures of prices of such credit derivatives.

The aim of this paper is to analyse the joint distribution of corporate times to default, focusing in particular on default correlation and on its implications for the term structure of corporate bonds and credit derivatives. It is an extension of the analysis performed by Jarrow, Yu (2001), and Schonbucher, Schubert (2001) [2]. Section 2 is concerned with the patterns of the term structures of corporate bonds according to the firms, which are still alive. The term structures can be written in terms of the joint survivor function of times to default, or in terms of default intensities. It is proved that they are sufficient to recover the price of any credit derivative, even with a payoff written jointly on several times to default. Moreover it is proved that the term structure of corporate interest rate will feature jumps whenever default correlation exists. Section 3 is concerned with factor models. We first consider models with a single time invariant factor. This factor represents unobserved individual heterogeneity and creates a default correlation characterized by an Archimedean copula. In this framework the corporate bond prices, the spreads of interest rates and the jump in intensities can be interpreted from the heterogeneity (factor) distribution. The analysis is successively extended to dynamic factor models. We discuss especially the role of the information set, and explain how intensities, jumps in the intensity, and default correlation depend on the selected information set. Section 4 concludes.

2 Characterisations of the spread term structures.

Let us consider two firms, with times to default Y_1 and Y_2, respectively. The joint survivor function of durations Y_1, Y_2 under the risk neutral distribution is denoted by:

$$S(y_1, y_2) = P[Y_1 \geq y_1, Y_2 \geq y_2].$$

[2] See also Yu (2003), who provides an algorithm for simulating correlated defaults with general dependence structure.

If default is independent of the riskfree term structure under the risk neutral probability, the price at t of a zero-coupon bond with residual maturity h associated with firm 1 is:

$$B_1(t,h) = B(t,h)\, P(Y_1 > t+h \mid I_t), \tag{2}$$

where $B(t,h)$ is the price of the riskfree zero-coupon bond and I_t denotes the information available at t. Formula (2) allows for a separate analysis of default and riskfree term structure [see e.g. Fons (1994)]. It is important to keep in mind that the conditional default probability $P(Y_1 > t+h \mid I_t)$ can be generally interpreted as a ratio of two prices of zero-coupon bonds. Similarly:

$$-\frac{1}{h}\log\frac{B_1(t,h)}{B(t,h)} = -\frac{1}{h}\log P(Y_1 > t+h \mid I_t)$$

is the spread of interest rates at term h. Without loss of generality, we assume a zero riskfree rate: $B(t,h) = 1, \forall t, h$, and thus systematically interpret $B_1(t,h)$ as a spread of prices.

2.1 The term structures of corporate bonds.

In this section we make the following assumption on the information set I_t.

Assumption A.1: The information I_t available to price future default includes the default history of the firms only.

This assumption has been adopted in several studies in the literature [see e.g. Bremaud (1981), Duffie, Singleton (1998), Jarrow, Yu (2001), Yu (2003)]. It is implicitly assumed that additional macrofactors, such as the short term riskfree interest rate or the market return, have no influence on default prices. This assumption will be relaxed in Section 3 concerning factor models.

To derive the prices of zero-coupon bonds issued by the two firms, we distinguish two cases according to the available information I_t. Indeed at date t both firms can be still alive, or one firm can have defaulted earlier.

i) Both firms are still alive

Let us consider the zero-coupon bonds issued by firm 1. At time t the price of this bond with residual maturity h when both firms are still alive is given by:

$$B_1(t,h) = P[Y_1 > t+h \mid Y_1 > t, Y_2 > t] = \frac{S(t+h,t)}{S(t,t)}, \quad \forall t, h. \tag{3}$$

ii) One firm defaulted earlier

If firm 2 defaulted at a previous date, $t-k$, say, the price at time t of the same bond is given by [see Appendix 1]:

$$B_1(t,h,k) = P\left[Y_1 > t+h \mid Y_1 > t, Y_2 = t-k\right] = \frac{\frac{\partial S}{\partial y_2}(t+h, t-k)}{\frac{\partial S}{\partial y_2}(t, t-k)}, \quad \forall t, h, \; k \leq t. \tag{4}$$

Similarly, the prices of zero coupon bonds issued by firm 2 are given by:

$$B_2(t,h) = \frac{S(t,t+h)}{S(t,t)}, \tag{5}$$

if firm 1 is still alive at t,

$$B_2(t,h,k) = \frac{\frac{\partial S}{\partial y_1}(t-k, t+h)}{\frac{\partial S}{\partial y_1}(t-k, t)}, \tag{6}$$

if firm 1 defaulted at time $t-k$.

Thus at time t the term structure associated with firm 1 depends on the situation of the second firm. If firm 2 is still alive it is given by $h \rightarrow B_1(t,h)$. If firm 2 defaulted at $t-k$, it is given by: $h \rightarrow B_1(t,h,k)$. Generally there exists a discontinuity of the term structure of interest rate spread according to the situation of firm 2, since:

$$\lim_{k \rightarrow 0} r_1(t,h,k) = r_1\left(t,h,0^+\right) \neq r_1(t,h),$$

where:

$$r_1(t,h) = -\frac{1}{h}\log B_1(t,h), \quad r_1(t,h,k) = -\frac{1}{h}\log B_1(t,h,k),$$

denote the geometric interest rate spreads. This discontinuity originates from the effect of default of firm 2 on the information set which is relevant for predicting default of firm 1. Under default independence the term structures feature no discontinuity.

When the two curves $h \rightarrow r_1(t,h,0^+), r_1(t,h)$ are different, they can differ for all terms [see Figure 1, Panel A], or simply for some terms. For instance they can differ in the long term, but coincide in the short term: $r_1(t,0,0^+) = r_1(t,0)$; in this case default of the second firm as no immediate effect on default intensity of firm 1 [see Figure 1, Panel B]. Alternatively, the two curves can differ in the short term, but can coincide in the long term: $r_1(t,\infty,0^+) = r_1(t,\infty)$; it will arise if the effect of default of firm 2 vanishes asymptotically [see Figure 1, Panel C].

Insert Figure 1A: Discontinuity of term structures

Insert Figure 1B: Continuity in the short term

Insert Figure 1C: Continuity in the long term

Finally the term structures coincide everywhere if and only if:

$$
\begin{aligned}
& \frac{S\,(t+h,t)}{S\,(t,t)} = \frac{\frac{\partial S}{\partial y_2}\,(t+h,t)}{\frac{\partial S}{\partial y_2}\,(t,t)}, \qquad \forall t, h \geq 0, \\
\Longleftrightarrow \quad & \frac{\partial \log S}{\partial y_2}\,(t+h,t) = \frac{\partial \log S}{\partial y_2}\,(t,t)\,, \qquad \forall t, h \geq 0, \\
\Longleftrightarrow \quad & \frac{\partial \log S}{\partial y_2}\,(y_1, y_2) \text{ is independent of } y_1, \text{ when } y_1 \geq y_2, \\
\Longleftrightarrow \quad & \text{the joint survivor function can be decomposed as a product:} \\
& \qquad S(y_1, y_2) = a(y_1) b(y_2), \text{ say, for } y_1 \geq y_2. \qquad (7)
\end{aligned}
$$

This condition can be seen as a type of independence condition on the cone $\{y_1 \geq y_2\}$ [3].

It is interesting to compare the term structures when condition (7) is satisfied [see Gouriéroux, Monfort (2003)]. We get:

$$
B_1(t,h,k) = \frac{\frac{\partial S}{\partial y_2}\,(t+h, t-k)}{\frac{\partial S}{\partial y_2}\,(t, t-k)} = \frac{a(t+h)}{a(t)} = B_1(t,h) \quad \forall t, h, k,
$$

and deduce the proposition below.

Proposition 1 *The term structure of firm 1 is continuous when firm* 2 *defaults if and only if it is independent of the situation of firm* 2*:*

$$
\begin{aligned}
& B_1(t,h,0^+) = B_1(t,h), \qquad \forall t, h \ , \\
\Longleftrightarrow \quad & B_1(t,h,k) = B_1(t,h), \qquad \forall t, h, k.
\end{aligned}
$$

In fact condition (7) can be seen as a noncausality condition from Y_2 to Y_1 [see Florens, Fougere (1996) for causality analysis of point processes], which explains the result of Proposition 1.

2.2 Equivalence between the marginal term structures and the joint distribution of default.

Equations (3)-(6) explain how to derive the marginal term structures from the joint survivor function of default. In this section we show that the marginal term structures actually provide an information equivalent to the joint survivor function. Note that generally it is not possible to deduce the price of a derivative with payoff written on two assets Y_1, Y_2, from the price of derivatives written on Y_1 only and of derivatives written on Y_2 only. In the case of default risk the situation can be different since the default of a firm can imply a jump in the derivative prices written on default of the other firm (due to the jump in the information set), revealing the dependence structure.

[3] More precisely the times to default are independent conditionally to $Y_1 > y_1, Y_2 < y_2$, for any y_1, y_2 with $y_1 > y_2$.

We can deduce from the term structure of firm 1 the following default intensities [see Cox, Oakes (1984) chap. 10] [4]:

$$
\begin{aligned}
\lambda_1(t) &\equiv \lim_{dt\to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 > t\right] \\
&= \lim_{dt\to 0} \frac{1-B_1(t,dt)}{dt} = -\frac{\partial \log S}{\partial y_1}(t,t), \qquad (8)\\
\gamma_1(t,t-k) &\equiv \lim_{dt\to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 = t-k\right] \\
&= \lim_{dt\to 0} \frac{1-B_1(t,dt,k)}{dt} = -\frac{\partial}{\partial y_1} \log\left[-\frac{\partial S}{\partial y_2}(t,t-k)\right]. \qquad (9)
\end{aligned}
$$

Function λ_1 is the default intensity for firm 1 at time t when both firms are still alive, and function $t \to \gamma_1(t,t-k)$ when firm 2 has defaulted at the previous date $t-k$. Function λ_1 corresponds to the short term spread at time t associated to firm 1 when both firms are still alive, since this short term spread is:

$$
\begin{aligned}
\lim_{dt\to 0} -\frac{1}{dt} \log\left[B_1(t,dt)/B(t,dt)\right] &= \lim_{dt\to 0} -\frac{1}{dt} \log B_1(t,dt) \\
&= \lim_{dt\to 0} \frac{1-B_1(t,dt)}{dt}.
\end{aligned}
$$

Similarly $\gamma_1(t,t-k)$ is the short term spread of firm 1 when the second firm defaulted earlier.

The intensity can feature a jump when one firm defaults. Let us assume that the joint survivor function admits a cross second order derivative on the diagonal. The sign and size of the jump are obtained by considering the difference:

$$
\begin{aligned}
\gamma_1\left(t,t^-\right) - \lambda_1(t) &= \frac{\partial \log S}{\partial y_1}(t,t) - \frac{\partial}{\partial y_1} \log\left[-\frac{\partial S}{\partial y_2}(t,t)\right] \\
&= -\frac{\partial}{\partial y_1} \log\left[-\frac{\partial \log S}{\partial y_2}(t,t)\right].
\end{aligned}
$$

In particular the jump is always positive if and only if:

$$
-\frac{\partial}{\partial y_1} \log\left[-\frac{\partial \log S}{\partial y_2}(t,t)\right] \geq 0, \quad \forall t.
$$

[4] Intensities $\lambda_2(t)$ and $\gamma_2(t,t-k)$ for firm 2 are defined similarly:

$$
\begin{aligned}
\lambda_2(t) &\equiv \lim_{dt\to 0} \frac{1}{dt} P\left[Y_2 < t+dt \mid Y_1 > t, Y_2 > t\right] \\
&= \lim_{dt\to 0} \frac{1-B_2(t,dt)}{dt} = -\frac{\partial \log S}{\partial y_2}(t,t), \\
\gamma_2(t,t-k) &\equiv \lim_{dt\to 0} \frac{1}{dt} P\left[Y_2 < t+dt \mid Y_2 > t, Y_1 = t-k\right] \\
&= \lim_{dt\to 0} \frac{1-B_2(t,dt,k)}{dt} = -\frac{\partial}{\partial y_2} \log\left[-\frac{\partial S}{\partial y_1}(t-k,t)\right].
\end{aligned}
$$

This condition is equivalent to:

$$\frac{\partial^2 \log S}{\partial y_1 \partial y_2}(t,t) \geq 0, \;\; \forall t,$$

or to:

$$\begin{aligned} & \frac{1}{S(t,t)} \frac{\partial^2 S}{\partial y_1 \partial y_2}(t,t) - \frac{1}{S(t,t)^2} \frac{\partial S}{\partial y_1}(t,t) \frac{\partial S}{\partial y_2}(t,t) \\ = \;& \lim_{dt \to 0} \frac{1}{dt^2} Cov\left(\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t, Y_2 > t\right) \geq 0, \;\; \forall t. \end{aligned} \tag{10}$$

Proposition 2 *If S is twice differentiable on the diagonal, the intensity jump is always nonnegative if and only if the infinitesimal default occurrences are positively correlated.*

Thus jumps in the term structure of credit spreads could be explained by default correlation[5].

The existence of the cross derivative of the joint survivor function implies restrictions on the intensity functions. More precisely it is easily checked that the cross derivative $\partial^2 S(t,t) / \partial y_1 \partial y_2$ exists if and only if $\lambda_1(t)/\gamma_1(t,t) = \lambda_2(t)/\gamma_2(t,t)$. Thus under this condition the jumps in intensity have the same relative sizes for both firms. When the cross order derivative does not exist, it is necessary to introduce two definitions of infinitesimal default covariance [see Appendix 2 i)]:

$$\lim_{dt \to 0} \frac{1}{dt^2} Cov\left(\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t-dt<Y_2<t} \mid Y_1 > t, Y_2 > t-dt\right), \tag{11}$$

$$\lim_{dt \to 0} \frac{1}{dt^2} Cov\left(\mathbb{I}_{t-dt<Y_1<t}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t-dt, Y_2 > t\right), \tag{12}$$

to point out the asymmetric reaction of default intensity of firm 1 to default of firm 2, and of default of firm 2 to default of firm 1, respectively. In this case the intensity jump of firm 1 [of firm 2, respectively] is always positive if and only if the instantaneous default correlation in expression (11) [expression (12), respectively] is positive.

The next proposition[6] is proved in Section 2.5.2, where its pricing interpretation is also discussed.

Proposition 3 *The knowledge of the intensities $\lambda_1, \gamma_1, \lambda_2, \gamma_2$ is equivalent to*

[5] See Zhou (2001) for term structure models with jumps, and Schonbucher, Schubert (2001) for an analysis of intensity jumps in a specific model of correlated defaults.

[6] See also Cox (1972), Cox, Lewis (1972), Griffiths, Milne (1978), Cox, Oakes (1984) chap. 10 for general presentations of bivariate duration models.

the knowledge of the joint survivor function. More precisely we have:

$$\begin{aligned} S\left(y_1, y_2\right) &= \exp\left[-\Lambda_1\left(y_1\right) - \Lambda_2\left(y_1\right)\right] + \int_{y_2}^{y_1} \lambda_2\left(y\right) e^{-\Lambda_1(y)-\Lambda_2(y)} e^{-\Gamma_1(y_1,y)} dy, \\ \textit{if} \quad y_1 &\geq y_2, \textit{ and} \\ S\left(y_1, y_2\right) &= \exp\left[-\Lambda_1\left(y_2\right) - \Lambda_2\left(y_2\right)\right] + \int_{y_1}^{y_2} \lambda_1\left(y\right) e^{-\Lambda_1(y)-\Lambda_2(y)} e^{-\Gamma_2(y_2,y)} dy, \\ \textit{if}\, y_1 &< y_2, \end{aligned}$$

where $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2$ *denote the integrated intensities:*

$$\Lambda_i(y) = \int_0^y \lambda_i(s) ds, \qquad \Gamma_i\left(z, y\right) = \int_y^z \gamma_i\left(s, y\right) ds, \;\; i = 1, 2.$$

Therefore the knowledge of the marginal term structures is equivalent to the knowledge of the joint survivor function S. This implies that the marginal term structures provide full information not only on the marginal distribution of the times to default Y_1, Y_2, but also on their dependence structure, that is default correlation. In particular we deduce the following corollary.

Corollary 4 *Under default correlation, there is a one-to-one relationship between the term structures of zero-coupon bonds, the short term spreads, and the joint survivor function.*

In fact it is possible to improve the equivalence given in Corollary 4. Indeed from equation (8), we can derive default intensities λ_1, λ_2 from the term structures $B_1(t,h)$, $B_2(t,h)$, $\forall t, h$. Moreover we have:

$$\begin{aligned} \lambda_1(t) + \lambda_2(t) &= -\frac{\partial \log S}{\partial y_1}(t,t) - \frac{\partial \log S}{\partial y_2}(t,t) \\ &= -\frac{d}{dt} \log S(t,t), \end{aligned}$$

and by integration:

$$S(t,t) = \exp\left[-\Lambda_1(t) - \Lambda_2(t)\right]. \tag{13}$$

Thus $S(t,t)$ can be computed from $B_1(t,h)$, $B_2(t,h)$, and also $S(t+h,t) = B_1(t,h)S(t,t)$ and $S(t,t+h) = B_2(t,h)S(t,t)$ can be:

$$S(t+h,t) = B_1(t,h) \exp\left[-\Lambda_1(t) - \Lambda_2(t)\right], \tag{14}$$

$$S(t,t+h) = B_2(t,h) \exp\left[-\Lambda_1(t) - \Lambda_2(t)\right]. \tag{15}$$

We deduce the following Corollary.

Corollary 5 *Under default correlation, there is a one-to-one relationship between the term structures of zero-coupon bonds computed when both firms are still alive, the short term spreads, and the joint survivor function.*

Finally from Proposition 3 and equations (3)-(6) we deduce the prices of bonds in terms of the intensities.

Corollary 6 *The prices of bonds issued by firm 1 are given by:*

$$\begin{aligned} B_1(t,h) &= e^{-[\Lambda_1(t+h)-\Lambda_1(t)]-[\Lambda_2(t+h)-\Lambda_2(t)]} \\ &\quad + \int_t^{t+h} \lambda_2(y)\, e^{-[\Lambda_1(y)-\Lambda_1(t)]-[\Lambda_2(y)-\Lambda_2(t)]-\Gamma_1(t+h-y,y)} dy, \\ B_1(t,h,k) &= \exp\left[-\int_t^{t+h} \gamma_1(s,t-k)ds\right]. \end{aligned}$$

When both firms are still alive, the price of the bond admits an exponential expression:

$$B_1(t,h) = \exp - \int_t^{t+h} \mu(s,t)ds,$$

where: $\mu(s,t) = \lim_{dt\to 0} P\left[Y_1 > t+s+dt \mid Y_1 > t+s, Y_2 > t\right]/dt$ depends generally on t, and in particular does not coincide with $\lambda_1(s)$.

2.3 Comparison with the approach by Jarrow, Yu (2001).

In Section 2.2 the bivariate duration model has been defined by means of the intensities $\lambda_1, \lambda_2, \gamma_1, \gamma_2$. Jarrow, Yu (2001), Section I, proposed to define the distribution by means of conditional intensities. Typically they consider the conditional distribution of Y_1 given Y_2 (resp. Y_2 given Y_1) and the associated hazard functions $\lambda(y_1 \mid y_2)$ [resp. $\lambda(y_2 \mid y_1)$]:

$$\lambda(y_1 \mid y_2) = \lim_{dt\to 0} \frac{1}{dt} P\left[Y_1 < y_1 + dt \mid Y_1 > y_1, Y_2 = y_2\right].$$

These intensities differ from λ_1 or γ_1 by the information set. Note in particular that y_2 can be larger than y_1.

Of course the conditional distribution can be derived from the conditional intensity:

$$P\left[Y_1 > y_1 \mid Y_2 = y_2\right] = S(y_1 \mid y_2) = \exp -\Lambda(y_1 \mid y_2),$$

where the cumulated conditional hazard is $\Lambda(y_1 \mid y_2) = \int_0^{y_1} \lambda(y \mid y_2)\, dy$. Moreover the knowledge of both conditional survivor functions $S(y_1 \mid y_2)$ and $S(y_2 \mid y_1)$ define unambiguously the joint distribution of (Y_1, Y_2) [see Gouriéroux, Monfort (1979)]. However it is also known that both conditional distributions cannot be chosen arbitrarily. They have to satisfy some compatibility restrictions [see Gouriéroux, Monfort (1979)]. More precisely the joint pdf can be compute from the conditional pdf by:

$$\begin{aligned} f(y_1, y_2) &= \frac{f(y_2 \mid y_1)}{\int \frac{f(y_2|y_1)}{f(y_1|y_2)} dy_2} \\ &= \frac{f(y_1 \mid y_2)}{\int \frac{f(y_1|y_2)}{f(y_2|y_1)} dy_1}. \end{aligned} \tag{16}$$

The second equality, which has to be satisfied for any y_1, y_2, defines the compatibility restrictions. Therefore the conditional intensities are compatible if and only if:

$$\begin{aligned}
& \lambda\left(y_2 \mid y_1\right) \frac{\exp -\Lambda\left(y_2 \mid y_1\right)}{\int \frac{\lambda\left(y_2 \mid y_1\right) \exp -\Lambda\left(y_2 \mid y_1\right)}{\lambda\left(y_1 \mid y_2\right) \exp -\Lambda\left(y_1 \mid y_2\right)} d y_2} \\
= \; & \lambda\left(y_1 \mid y_2\right) \frac{\exp -\Lambda\left(y_1 \mid y_2\right)}{\int \frac{\lambda\left(y_1 \mid y_2\right) \exp -\Lambda\left(y_1 \mid y_2\right)}{\lambda\left(y_2 \mid y_1\right) \exp -\Lambda\left(y_2 \mid y_1\right)} d y_1}, \quad \forall y_1, y_2 .
\end{aligned} \tag{17}$$

Jarrow, Yu (2001) proposed a specification of the type (p 1772):

$$\begin{aligned}
\lambda\left(y_1 \mid y_2\right) &= a_1+a_2 \mathbb{I}_{y_1>y_2}, \\
\lambda\left(y_2 \mid y_1\right) &= b_1+b_2 \mathbb{I}_{y_2>y_1} .
\end{aligned} \tag{18}$$

Their justification is "when firm 2 defaults, firm 1's default probability will jump and vice-versa". In fact this idea has to be written on λ_1, γ_1 as described in previous sections, not on the conditional intensities. As a consequence the parameters in (18) cannot be chosen arbitrarily. It is easy (but cumbersome) to check that the compatibility restriction implies:

$$a_2=0 \text{ or } b_2=0,$$

that is a recursive specification. In fact this is essentially the case completely studied by Jarrow, Yu (2001). Finally note that, under the compatibility restriction, the joint distribution is easily derived from (16), (17).

2.4 Examples

In order to illustrate the results above, let us discuss several examples.

Example 1: Constant intensities.

Let us assume constant intensities given by:

$$\lambda_1(t)=r_1, \quad \gamma_1(t, t-k)=r_1^*, \quad \lambda_2(t)=r_2, \quad \gamma_2(t, t-k)=r_2^* .$$

The joint survivor function becomes:

$$S\left(y_1, y_2\right)=\exp \left[-\left(r_1+r_2\right) y_1\right]+r_2 e^{-r_1^* y_1} \int_{y_2}^{y_1} e^{-\left[r_1+r_2-r_1^*\right] y} d y, \quad \text{if } y_1>y_2 .$$

i) If $r_1^* \neq r_1+r_2$, we get:

$$S\left(y_1, y_2\right)=\frac{r_1-r_1^*}{r_1+r_2-r_1^*} e^{-\left(r_1+r_2\right) y_1}+\frac{r_2}{r_1+r_2-r_1^*} e^{-r_1^* y_1} e^{-\left(r_1+r_2-r_1^*\right) y_2}, \quad \text{for } y_1>y_2 .$$

The prices of bonds issued by firm 1 are given by:

$$\begin{aligned} B_1(t,h) &= \frac{r_1 - r_1^*}{r_1 + r_2 - r_1^*} e^{-(r_1+r_2)h} + \frac{r_2}{r_1 + r_2 - r_1^*} e^{-r_1^* h}, \text{ if firm 2 is still alive at } t, \\ B_1(t,h,k) &= e^{-r_1^* h}, \qquad \text{if firm 2 defaulted at } t-k. \end{aligned}$$

This term structure is meaningful only when firm 1 is still alive at date t. This explains why the zero-coupon prices are independent of r_2^*. Moreover the long term spread is given by $\min\{r_1 + r_2, r_1^*\}$.

We provide in Figure 2, Panel A, the term structure associated with firm 1 when both firms are still alive, for parameters $r_1 = 0.01$, $r_2 = 0.02$, and different values of r_1^*.

Insert Figure 2A: constant intensities: term structure when both firms are still alive

This term structure is constant in time. Moreover it is increasing (decreasing) in the maturity when $r_1^* > r_1$ [$r_1^* < r_1$], that is when the occurrence of default of the second firm increases (decreases, respectively) the default intensity of the first firm. When $r_1^* = r_1$, default intensity of firm 1 is independent of the situation of firm 2, and the term structure is flat. In Panel B we provide the term structure of firm 1 when the second firm defaulted earlier.

Insert Figure 2B: constant intensities: term structure when firm 2 defaulted earlier

This term structure is flat at level $r_1^* = 0.05$, and constant in time. The short term spreads for firm 1 are reported in Panels C and D.

Insert Figure 2C: constant intensities: short term spread when $Y_2 > Y_1$

Insert Figure 2D: constant intensities: short term spread when $Y_2 < Y_1$

In Panel C the second firm defaults after firm 1 [$Y_2 > Y_1 = 7$], and the short term spread is constant at r_1. In Panel D, the default of firm 2 occurs before, $Y_2 = 4$, and at that date the short term spread of firm 1 jumps at $r_1^* = 0.05$. Finally, the interest rate spreads for a zero-coupon bond issued by firm 1 and with given maturity $t = H = 10$ are reported in Figure 3, Panels A and B, in the case where firm 2 defaults after maturity H [respectively, before with $Y_2 = 7$].

Insert Figure 3A: constant intensities: spread for a fixed maturity H when $Y_2 > H$

Insert Figure 3B: constant intensities: spread for a fixed maturity H when $Y_2 < H$

In the first case the spread is decreasing in time, and takes the value $r_1 = 0.01$ at maturity; in the second case it features a jump at time to default of firm 2, and is constant at $r_1^* = 0.05$ afterwards.

ii) If $r_1^* = r_1 + r_2$, the two possible values of the long term spread coincide. The joint survivor function becomes:

$$S(y_1, y_2) = [1 + r_2(y_1 - y_2)] e^{-(r_1+r_2)y_1}, \quad \text{for } y_1 > y_2,$$

and the prices of bonds issued by firm 1 are given by:

$$\begin{aligned} B_1(t,h) &= [1+r_2 h]\, e^{-(r_1+r_2)h}, && \text{if firm 2 is still alive at } t, \\ B_1(t,h,k) &= e^{-r_1^* h}, && \text{if firm 2 defaulted at } t-k. \end{aligned}$$

The marginal survivor function of y_1 becomes:

$$S_1(y_1) = \frac{r_1 - r_1^*}{r_1 + r_2 - r_1^*} e^{-(r_1+r_2)y_1} + \frac{r_2}{r_1 + r_2 - r_1^*} e^{-r_1^* y_1},$$

which is a mixture of two exponential distributions, with parameters $r_1 + r_2$ and r_1^*; they correspond to the intensity of $\min(Y_1, Y_2)$ and to the intensity of Y_1 given $Y_2 = y, Y_1 > y$, respectively.

The conditional hazard function of Y_1 given $Y_2 = y_2$ is given by:

$$\lambda_1(y_1 \mid y_2) = r_1 \frac{r_2^*}{r_2 e^{-\left(r_1+r_2-r_2^*\right)(y_2-y_1)} + \frac{r_1 r_2^*}{r_1+r_2-r_2^*}\left(1 - e^{-\left(r_1+r_2-r_2^*\right)(y_2-y_1)}\right)},$$

if $y_1 < y_2$, and:

$$\lambda_1(y_1 \mid y_2) = r_1^*,$$

if $y_1 \geq y_2$.

Contrary to a natural belief the conditional hazard function $\lambda_1(y_1 \mid y_2)$ is not a stepwise function: $\lambda_1(y_1 \mid y_2) = \lambda_1 \mathbb{I}_{y_1 \leq y_2} + \lambda_1^* \mathbb{I}_{y_1 \geq y_2}$, say, when the underlying intensities are constant. This type of condition has been introduced for instance in Jarrow, Yu (2001), Section I. The stepwise condition on the conditional hazard function is satisfied if and only if $r_2 = r_2^*$. This condition is a noncausality condition from Y_1 to Y_2 [see Florens, Fougere (1996)]. The term structures of the corporate bonds have been studied analytically by Jarrow, Yu (2001) in this special case only [see the discussion in Section 2.3; see also Bielecki, Rutkowski (2002), Chapter 10, and Yu (2003).].

Example 2: Model with proportional hazard

Let us consider the extension of Example 1 characterized by the following intensities:

$$\begin{aligned} \lambda_1(t) &= r_1 \lambda_0(t), & \lambda_2(t) &= r_2 \lambda_0(t) \\ \gamma_1(t, t-k) &= r_1^* \lambda_0(t), & \gamma_2(t, t-k) &= r_2^* \lambda_0(t), \end{aligned}$$

where λ_0 is a positive function. Thus all intensities are proportional to a same baseline hazard function. In particular, the intensities can depend on the default occurrence of the other firm, but the intensities γ_1 and γ_2 are independent of the date of default of the other firm. In fact this model is equivalent to the model of Example 1 after applying an appropriate time deformation. More precisely it is easily checked that the transformed variables $\Lambda_0(Y_1), \Lambda_0(Y_2)$ admit constant intensities. The survivor function is deduced from the survivor function of Example 1 by replacing y_i by $\Lambda_0(y_i)$, $i = 1, 2$. For instance if $r_1^* \neq r_1 + r_2$, we get:

$$S(y_1, y_2) = \frac{r_1 - r_1^*}{r_1 + r_2 - r_1^*} e^{-(r_1+r_2)\Lambda_0(y_1)} + \frac{r_2}{r_1 + r_2 - r_1^*} e^{-r_1^* \Lambda_0(y_1)} e^{-(r_1+r_2-r_1^*)\Lambda_0(y_2)},$$

for $y_1 > y_2$, and:

$$\begin{aligned} B_1(t,h) &= \frac{r_1 - r_1^*}{r_1 + r_2 - r_1^*} e^{-(r_1+r_2)[\Lambda_0(t+h)-\Lambda_0(t)]} + \frac{r_2}{r_1 + r_2 - r_1^*} e^{-r_1^*[\Lambda_0(t+h)-\Lambda_0(t)]}, \\ &\quad \text{if firm 2 is still alive at } t, \\ B_1(t,h,k) &= e^{-r_1^*[\Lambda_0(t+h)-\Lambda_0(t)]}, \qquad \text{if firm 2 defaulted at } t-k. \end{aligned}$$

The price of the zero-coupon bond now depends on both date t and residual maturity h, contrary to the special case described in Example 1. Moreover the price specification is semi-nonparametric, with functional parameter Λ_0 and scalar parameters r_1, r_2, r_1^*, r_2^*. In particular, by appropriate choices of the time transformation, we can reproduce the spread patterns observed in practice, which are typically hump-shaped. We provide in Figure 4, Panel A, the term structure associated with firm 1 at time $t = 1$, when both firms are still alive, for the parameters $r_1 = 0.01, r_2 = 0.02$, a baseline hazard $\lambda_0(t) = 1/(1+t)^{0.3}$, and different values of r_1^*.

Insert Figure 4A: proportional hazard: term structure when both firms are still alive

The pattern of the term structure is affected both by the jump in the intensity according to the situation of firm 2 and by the shape of the baseline hazard λ_0. For our parameter choice the latter is decreasing, and when r_1^* is sufficiently larger than r_1, the term structure is hump-shaped. The hump arises when the jump of the intensity is sufficiently large compared to the decreasing effect of the baseline hazard function. Of course there exist other ways for creating such a hump, for instance by selecting a baseline hazard with hump. In Panel B we provide the term structure associated with firm 1 at time $t = 1$ when firm 2 defaulted earlier.

Insert Figure 4B: proportional hazard: term structure when firm 2 defaulted earlier

It is decreasing since the baseline hazard function λ_0 is. Finally, the short term spreads for firm 1 are reported in Panels C and D, when firm 2 defaults after firm 1 [$Y_2 > Y_1$], and when firm 2 defaults before firm 1 [$Y_1 = 7$, $Y_2 = 4$], respectively.

Insert Figure 4C: proportional hazard: short term spread when $Y_2 > Y_1$

Insert Figure 4D: proportional hazard: short term spread when $Y_2 < Y_1$

In the second case the short term spread features a jump at the time firm 2 defaults.

Example 3: Flat term structures

The term structures are flat when both firms are still alive, if:

$$B_1(t,h) = \exp\left[-\lambda_1(t)h\right], \qquad B_2(t,h) = \exp\left[-\lambda_2(t)h\right].$$

The corporate interest rate spreads $r_i(t,h) = \lambda_i(t)$, $i = 1, 2$, are independent of the maturity h, but they can depend on date t. From (14), (15) the joint survivor function becomes:

$$S(y_1, y_2) = \exp\left[-\lambda_1(y_2)(y_1 - y_2) - \Lambda_1(y_2) - \Lambda_2(y_2)\right], \text{ if } y_1 \geq y_2,$$

and:

$$S(y_1, y_2) = \exp\left[-\lambda_2(y_1)(y_2 - y_1) - \Lambda_1(y_1) - \Lambda_2(y_1)\right], \text{ if } y_1 < y_2.$$

The short term spreads $\lambda_1(t)$, $\lambda_2(t)$ cannot be chosen arbitrarily [see Appendix 3]:

Proposition 7 *The joint survivor function is well-defined if and only if:*

$$\begin{aligned} 0 &\leq \frac{d\lambda_1(t)}{dt} \leq \lambda_1(t)\lambda_2(t), \\ 0 &\leq \frac{d\lambda_2(t)}{dt} \leq \lambda_1(t)\lambda_2(t). \end{aligned}$$

Thus the intensities λ_1 and λ_2 have to be increasing functions and their rate of increase cannot be too large. The condition for the joint survivor function to be well-defined is essentially:

$$\begin{aligned} \Delta &= S(y_1 + dy_1, y_2 + dy_2) - S(y_1, y_2 + dy_2) - S(y_1 + dy_1, y_2) + S(y_1, y_2) \geq 0, \\ &\quad \forall y_1, y_2, dy_1, dy_2. \end{aligned}$$

Since Δ is the price of a credit derivative paying 1, if $y_1 < Y_1 < y_1 + dy_1$ and $y_2 < Y_2 < y_2 + dy_2$ (when the interest rate is zero), this condition is necessary for the absence of arbitrage opportunity among credit derivatives. This condition can be compared to the necessary condition on the long run riskfree interest rate implied by no arbitrage [El Karoui, Frachot, Geman (1998)]. Indeed the long run riskfree interest rate has to be an increasing function of time. Since $\lambda_i(t)$ is in particular the corporate long run interest rate, when the term structure is flat, it is not surprising to get the same type of conditions $[d\lambda_i/dt > 0]$, even for credit risky interest rates.

The intensity restrictions given in Proposition 7 are rather strong. Let us assume for instance that firm 1 is AAA with a constant intensity $\lambda_1(t) = \lambda_1$, close to zero. The set of restrictions reduces to $0 \leq d\log\lambda_2(t)/dt \leq \lambda_1$, $\forall t$, which limits the time dependence of λ_2.

Let us now derive the intensities when one firm has defaulted. We have [see Appendix 3]:

$$-\frac{\partial S}{\partial y_2}(y_1, y_2) = S(y_1, y_2)\left[\lambda_1'(y_2)(y_1 - y_2) + \lambda_2(y_2)\right],$$

and thus:

$$\gamma_1(t,t-k) = -\frac{\partial}{\partial y_1}\left[\log - \frac{\partial S}{\partial y_2}\right](t,t-k) = \lambda_1(t-k) - \frac{\lambda_1^{'}(t-k)}{\lambda_1^{'}(t-k)k + \lambda_2(t-k)}.$$

When the time to default $t-k$ of firm 2 is given, the intensity γ_1 is an increasing function of k. The intensity takes value $\lambda_1(t-k)$ just before default of firm 2, value $\lambda_1(t-k) - \lambda_1^{'}(t-k)/\lambda_2(t-k)$ just after default of firm 2, and value $\lambda_1(t-k)$ when k is infinite. In particular the conditions of Proposition 7 ensure the positivity of intensity γ_1, and imply that the intensity jump at default time is necessarily nonpositive, which correspond to negative default correlation. The term structures when both firms are still alive are flat since the effect of increasing default intensity and negative default correlation exactly compensate. Finally, the asymptotic behaviour corresponds to an intensity reverting phenomenon: the shock due to default of firm 2 has no persistent effect.

From Corollary 6 we deduce the term structure of firm 1 when the second firm has defaulted at time $t-k$ [see Appendix 3]:

$$B_1(t,h,k) = \exp\left(-h\left[\lambda_1(t-k) - \frac{1}{h}\log\left(1 + \frac{\lambda_1^{'}(t-k)}{\lambda_1^{'}(t-k)k + \lambda_2(t-k)}h\right)\right]\right).$$

As an illustration we provide the term structures and the short term spreads when the intensities λ_i are given by $\lambda_i(t) = r_i \exp(\beta_i t)$, with $r_1 = 0.01$, $r_2 = 0.05$, $\beta_1 = 0.05$, $\beta_2 = 0.01$. In Figure 5, Panel A, we report the term structure of firm 1 at time $t = 4$ when both firms are still alive. This term structure is flat by assumption.

Insert Figure 5A: flat term structures: term structure when both firms are still alive

The term structure of firm 1 at date $t = 5$ when the second firm has defaulted at the previous date $t-k = 4$ is provided in Panel B.

Insert Figure 5B: flat term structures: term structure when firm 2 defaulted earlier

This term structure is increasing, and features a lower level compared to Panel A. Finally the short term spreads of firm 1 are reported in Panel C and Panel D, when the second firm defaults after, respectively before, $t = 10$.

Insert Figure 5C: flat term structures: short term spread when $Y_2 > 10$

Insert Figure 5D: flat term structures: short term spread when $Y_2 < 10$

In the second case, the short term spread of firm 1 features a negative jump at the date of default of the second firm [$Y_2 = 4$], and increases afterwards, reaching the pre-jump level asymptotically.

2.5 Credit derivatives

2.5.1 First-to-default basket

The values of the survivor function $S(y_1, y_2)$ can be considered as prices of derivatives jointly written on both times to default. Let us assume $y_1 \geq y_2$; then the derivative pays 1$ at date y_1 if $Y_1 \geq y_1$ and $Y_2 \geq y_2$. In particular when $y_1 = y_2$ we get a first-to-default basket.

Let us first study the first-to-default term structure. The price at time t of a first-to-default basket with residual maturity h is given by [see (13)]:

$$\begin{aligned} C(t,h) &= P[Y_1 > t+h, Y_2 > t+h \mid Y_1 > t, Y_2 > t] \\ &= \frac{S(t+h,t+h)}{S(t,t)} = \exp - \int_t^{t+h} [\lambda_1(u) + \lambda_2(u)]\, du. \end{aligned}$$

From Corollary 5 the first-to-default term structure is implied by the two marginal term structures of the firms computed when the two firms are alive. In particular the instantaneous interest rate associated with the first-to-default basket is:

$$r_c(t) = \lim_{h \to 0} -\frac{1}{h} \log C(t,h) = \lambda_1(t) + \lambda_2(t).$$

Thus this instantaneous interest rate is the sum of the instantaneous rates corresponding to both firms [see also Duffie (1998)]. This result is a consequence of an instantaneous independence between failures' occurrences. Indeed between t and $t + dt$, the default probabilities are:

$$\begin{aligned} P[Y_1 \leq t+dt, Y_2 \leq t+dt \mid Y_1 > t, Y_2 > t] &= o(dt), \\ P[Y_1 > t+dt, Y_2 \leq t+dt \mid Y_1 > t, Y_2 > t] &= \lambda_2(t)dt + o(dt), \\ P[Y_1 \leq t+dt, Y_2 > t+dt \mid Y_1 > t, Y_2 > t] &= \lambda_1(t)dt + o(dt), \\ P[Y_1 > t+dt, Y_2 > t+dt \mid Y_1 > t, Y_2 > t] &= 1 - \lambda_1(t)dt - \lambda_2(t)dt + o(dt). \end{aligned}$$

These conditions are equivalent to:

$$\begin{aligned} P[Y_1 \leq t+dt, Y_2 \leq t+dt \mid Y_1 > t, Y_2 > t] &= \lambda_1(t)\lambda_2(t)\,(dt)^2 + o(dt), \\ P[Y_1 > t+dt, Y_2 \leq t+dt \mid Y_1 > t, Y_2 > t] &= [1 - \lambda_1(t)dt]\,\lambda_2(t)dt + o(dt), \\ P[Y_1 \leq t+dt, Y_2 > t+dt \mid Y_1 > t, Y_2 > t] &= [1 - \lambda_2(t)dt]\,\lambda_1(t)dt + o(dt), \\ P[Y_1 > t+dt, Y_2 > t+dt \mid Y_1 > t, Y_2 > t] &= [1 - \lambda_1(t)dt]\,[1 - \lambda_2(t)dt] + o(dt), \end{aligned}$$

where the first components of the right hand side feature the independence property[7].

As an illustration, we provide below the term structure of the first-to-default basket for the examples considered in section 2.4.

Example 2 (continued):

[7] The default occurrences are independent at first order in dt. However dependence at second order in dt is introduced by default correlation, see equation (10).

For a model with proportional hazard: $S(t,t) = \exp\left[-(r_1+r_2)\Lambda_0(t)\right]$, and the term structure is given by $r_C(t,h) = (r_1+r_2)\left[\Lambda_0(t+h) - \Lambda_0(t)\right]/h$, whereas the first-to-default intensity is $r_C(t) = (r_1+r_2)\lambda_0(t)$. Therefore this intensity is also proportional to the baseline intensity.

Example 3 (continued):

For flat term structures we get: $S(t,t) = \exp\left[-\Lambda_1(t) - \Lambda_2(t)\right]$; the term structure of first-to-default prices is given by: $r_C(t,h) = \left[\Lambda_1(t+h) - \Lambda_1(t)\right]/h + \left[\Lambda_2(t+h) - \Lambda_2(t)\right]/h$.

2.5.2 Pricing interpretation of the joint survivor function

The interpretation in terms of first-to-default basket can be used to understand the expression of the survivor function in the general framework given in Proposition 3. Indeed let us assume $y_1 > y_2$; then:

$$\mathbb{I}_{Y_1>y_1,Y_2>y_2} = \mathbb{I}_{Y_1>y_1,Y_2>y_1} + \mathbb{I}_{Y_1>y_1}\mathbb{I}_{y_2<Y_2<y_1}. \tag{19}$$

The first term in the RHS of equation (19) is the payoff of a first-to-default basket with maturity y_1. Its current price is given by $\exp\left[-\Lambda_1(y_1) - \Lambda_2(y_1)\right]$. Therefore the price of the digital option paying 1\$ at date y_1 if $Y_1 > y_1$ and $y_2 < Y_2 < y_1$ [the second term in the RHS of equation (19)] must be equal to:

$$\begin{aligned} I &= \int_{y_2}^{y_1} \lambda_2(y)\, e^{-\Lambda_1(y)-\Lambda_2(y)} e^{-\Gamma_1(y_1,y)} dy \\ &= \int_{y_2}^{y_1} e^{-\Gamma_1(y_1,y)} \frac{\lambda_2(y)}{\lambda_1(y)+\lambda_2(y)} \left[\lambda_1(y)+\lambda_2(y)\right] e^{-\Lambda_1(y)-\Lambda_2(y)} dy. \end{aligned}$$

This decomposition is easily understood since[8]:

$$\begin{aligned} & E\left[\mathbb{I}_{Y_1>y_1}\mathbb{I}_{y_2<Y_2<y_1}\right] \\ = \; & E\left[E\left[\mathbb{I}_{Y_1>y_1}\mathbb{I}_{Y_1>Y_2} \mid \min(Y_1,Y_2) = y\right]\mathbb{I}_{y_2\le\min(Y_1,Y_2)\le y_1}\right] \\ = \; & E\left[P\left[Y_1>y_1 \mid \min(Y_1,Y_2)=y, Y_1>Y_2\right]\right. \\ & \left. P\left[Y_1>Y_2 \mid \min(Y_1,Y_2)=y\right]\mathbb{I}_{y_2\le\min(Y_1,Y_2)\le y_1}\right] \\ = \; & E\left[P\left[Y_1>y_1 \mid Y_2=y, Y_1>y\right] P\left[Y_1>Y_2 \mid \min(Y_1,Y_2)=y\right]\right. \\ & \left.\mathbb{I}_{y_2\le\min(Y_1,Y_2)\le y_1}\right], \end{aligned}$$

and:

$$\begin{aligned} P\left[Y_1>y_1 \mid Y_2=y, Y_1>y\right] &= \exp -\Gamma_1(y_1,y), \\ P\left[Y_1>Y_2 \mid \min(Y_1,Y_2)=y\right] &= \frac{\lambda_2(y)}{\lambda_1(y)+\lambda_2(y)}, \end{aligned}$$

whereas the density of the $\min(Y_1,Y_2)$ is $\left[\lambda_1(y)+\lambda_2(y)\right] e^{-\Lambda_1(y)-\Lambda_2(y)}$.

[8] Note that the interpretation below provides a proof of Proposition 3.

2.6 Extension to an arbitrary number of firms.

The previous results can be extended to an arbitrary number N of firms. Let us denote by $Y_1, Y_2, ..., Y_N$ the times to default, and by $S(y_1, ..., y_N)$ their joint survivor function. Again we have to condition on past default occurrences of the firms. For instance, the price at time t of a zero coupon bond of firm 1 with residual maturity h when firms 1 to m are still alive and firms $m+1, ..., n$ defaulted at $t - k_{m+1}, ..., t - k_n$, respectively, is given by:

$$\begin{aligned} & B_1(t, h, k_{m+1}, ..., k_n) \\ = \; & E\left[\mathbb{I}_{Y_1 > t+h} \mid Y_1 > t, ..., Y_m > t, Y_{m+1} = t - k_{m+1}, ..., Y_n = t - k_n\right] \\ = \; & \frac{\frac{\partial^{(n-m)}}{\partial y_{m+1} ... \partial y_n} S(t+h, t, ..., t, t - k_{m+1}, ..., t - k_n)}{\frac{\partial^{(n-m)}}{\partial y_{m+1} ... \partial y_n} S(t, t, ..., t, t - k_{m+1}, ..., t - k_n)}. \end{aligned} \tag{20}$$

The corresponding default intensities are given by:

$$\begin{aligned} & \gamma_1(t, t - k_{m+1}, ..., t - k_n) \\ = \; & \lim_{dt \to 0} \frac{1}{dt} P\left[Y_1 < t + dt \mid Y_1 > t, ..., Y_m > t, Y_{m+1} = t - k_{m+1}, ..., Y_n = t - k_n\right] \\ = \; & \lim_{dt \to 0} \frac{1 - B_1(t, dt, k_{m+1}, ..., k_n)}{dt}. \end{aligned} \tag{21}$$

3 Heterogeneity, jumps in spreads and default correlation

This section considers corporate bond pricing in factor models. In the first subsection the underlying factors are time independent. We first consider a model where the times to default depend on a single static factor, which allows to study in detail a default correlation represented by an Archimedean copula [see e.g. Genest, McKay (1986)]. The model is then extended to incorporate idiosyncratic static factors. We discuss the associated copula which extends the Archimedean copula and the associated corporate term structures. The second subsection considers dynamic factor models. We focus on the dependence of intensities, jumps in intensities and default correlations with respect to the selected information set.

The factor models are especially useful for homogeneous portfolios. For this reason we assume in this section the homogeneity condition, that is the symmetry of the joint distribution of times to default[9].

[9] Also called equidependence in the literature [see e.g. Frey, Mc Neil (2001), Gouriéroux, Monfort (2002)].

3.1 Static factor models.

3.1.1 A model with a common unobservable risk factor

i) Term structures

Let us assume that times to default Y_1 and Y_2 are independent conditionally to a positive factor Z, and follow exponential distributions $\gamma(1, Z)$ with constant intensity Z. By integrating factor Z, the joint survivor function of durations Y_1, Y_2 is given by:

$$S(y_1, y_2) = E\left[e^{-(y_1+y_2)Z}\right] = \Psi(y_1 + y_2),$$

where $\Psi = \exp -\psi$ denotes the Laplace transform of factor Z [10] and is cross-differentiable on the diagonal. This specification corresponds to the so-called Multivariate Mixed Proportional Hazard (MMPH) model [see e.g. Van den Berg (1997), (2001)], and to an Archimedean copula to characterize nonlinear dependence[11]. The factor Z can be seen as an unobserved heterogeneity factor with identical effects on corporate default intensities. This factor is independent of time.

Let us derive the term structure of corporate bonds [see Gouriéroux, Monfort (2003)]. From equations (3), (4) we get:

$$B_1(t, h) = \frac{\Psi(2t + h)}{\Psi(2t)}, \qquad B_1(t, h, k) = \frac{\Psi^{'}(2t + h - k)}{\Psi'(2t - k)}. \tag{22}$$

From (8), (9) the intensities are given by:

$$\lambda_1(t) = -\frac{\Psi^{'}(2t)}{\Psi(2t)} = \psi^{'}(2t), \tag{23}$$

$$\gamma_1(t, t - k) = -\frac{\Psi^{''}(2t - k)}{\Psi'(2t - k)} = \psi^{'}(2t - k) - \frac{\psi^{''}(2t - k)}{\psi^{'}(2t - k)}. \tag{24}$$

Note that the intensities are positive, since the Laplace transform of the heterogeneity distribution is necessarily decreasing, convex. The formulas above are easily interpreted. Indeed:

$$\begin{aligned} B_1(t, h) &= E\left[P(Y_1 > t + h \mid Y_1 > t, Y_2 > t, Z) \mid Y_1 > t, Y_2 > t\right] \\ &= E\left[\exp(-hZ) \mid Y_1 > t, Y_2 > t\right], \end{aligned} \tag{25}$$

[10] that is $\Psi(y) = E\left[\exp(-yZ)\right]$. Since Z is positive, the Laplace transform is defined for any nonnegative argument y and characterizes the distribution of Z.

[11] Indeed the survivor copula of this distribution [see e.g. Clayton (1978), Oakes (1982), Genest, McKay (1986), Joe (1997), Gagliardini, Gouriéroux (2002)] is: $C(u, v) = S\left[S_1^{-1}(u), S_2^{-1}(v)\right]$, where S_1, S_2 are the marginal survivor functions. Thus we get: $C(u, v) = \Psi\left[\Psi^{-1}(u) + \Psi^{-1}(v)\right]$, that is an Archimedean copula.

due to the lack of memory property of the exponential distribution, and similarly:

$$B_1(t,h,k) = E\left[\exp\left(-hZ\right) \mid Y_1 > t, Y_2 = t-k\right]. \tag{26}$$

Thus the term structure coincides with the Laplace transform of the factor Z conditionally to the available information I_t. Similar interpretations can be derived for default intensities. We get:

$$\lambda_1(t) = -\left.\frac{\partial B_1\left(t,h\right)}{\partial h}\right|_{h=0} = E\left[Z \mid Y_1 > t, Y_2 > t\right], \tag{27}$$

and similarly:

$$\gamma_1(t, t-k) = E\left[Z \mid Y_1 > t, Y_2 = t-k\right]; \tag{28}$$

thus the intensity is the expectation of factor Z given the available information I_t [12].

The interpretation of the term structure and intensities as conditional expectations with respect to the distribution of the factor Z given the available information I_t explains the patterns and the time evolution of the term structure and of the default intensities of a firm. For instance, since the zero-coupon prices coincide with values of a Laplace transform of a positive variable, the term structures of zero-coupon prices are decreasing, convex functions of h tending to zero in the long run. The rate of decay to zero depends on the heterogeneity distribution. The more concentrated the initial heterogeneity distribution, the smaller the rate of decay. Moreover, when time increases we get more information about default histories of both firms and we update our initial belief about the heterogeneity factor. As a consequence the distribution of factor Z is more concentrated, when t increases, at the lower bound of its support. In the long run the term structures of interest rates become flat, and tends to the lower bound of the support of the heterogeneity distribution. More precisely, we have the following Proposition [see Appendix 4].

Proposition 8 *Let times to default Y_1, Y_2 follow a MMPH model with heterogeneity factor Z, and let $z_1 \geq 0$ be the lower bound in the support of Z. Then:*

i) the term structures $h \to -\frac{1}{h}\log B_1(t,h), -\frac{1}{h}\log B_1(t,h,k)$ are decreasing;

ii) the long term spreads are equal to z_1, independent of time;

iii) for any term $h \geq 0$, the spreads $-\frac{1}{h}\log B_1(t,h), -\frac{1}{h}\log B_1(t,h,k)$ are decreasing functions of time t, and converge to z_1 when $t \to \infty$.

In particular, as time increases, the discontinuity in the term structure of firm 1 when firm 2 defaults becomes smaller. Asymptotically the default of firm 2 has no effect on the term structure of firm 1.

[12] Equations (27) and (28) correspond in this framework to the general formulas for the transformation of intensities under change of filtration in a point process [see e.g. Bremaud (1981), chapter II, Theorem 14, page 32].

In fact it is possible to say more on the term structure of the heterogeneity distribution [see Gouriéroux, Monfort (2003), Schonbucher (2003)]. Let us denote by $\Psi_t(y)$ [resp. $\Psi_{t,k}(y)$] the Laplace transform of the distribution of Z given $Y_1 > t, Y_2 > t$ [resp. $Y_1 > t, Y_2 = t-k$]. From (22), (25), (26) we get:

$$\Psi_t(h) = \frac{\Psi(2t+h)}{\Psi(2t)}, \qquad \Psi_{t,k}(h) = \frac{\Psi^{'}(2t+h-k)}{\Psi'(2t-k)},$$

Example 4: Discrete heterogeneity distribution

For an heterogeneity factor Z with discrete distribution:

$$Z = \begin{cases} z_1, & \text{with prob. } \pi, \\ z_2, & \text{with prob. } 1-\pi, \end{cases}$$

where $z_1 < z_2$, the Laplace transform is given by:

$$\Psi(y) = \pi \exp(-z_1 y) + (1-\pi)\exp(-z_2 y),$$

and the intensities of firm 1 are:

$$\begin{aligned} \lambda_1(t) &= z_1 \frac{1}{1+\frac{1-\pi}{\pi}e^{-2t\Delta z}} + z_2 \frac{\frac{1-\pi}{\pi}e^{-2t\Delta z}}{1+\frac{1-\pi}{\pi}e^{-2t\Delta z}}, \\ &\quad \text{when both firms are still alive,} \\ \gamma_1(t, t-k) &= z_1 \frac{1+\frac{1-\pi}{\pi}\left(\frac{z_2}{z_1}\right)^2 e^{-(2t-k)\Delta z}}{1+\frac{1-\pi}{\pi}\frac{z_2}{z_1}e^{-(2t-k)\Delta z}}, \\ &\quad \text{when firm 2 defaulted at } t-k, \end{aligned}$$

where $\Delta z = z_2 - z_1 > 0$. Intensity λ_1 is a weighted average of the two basic intensities z_1, z_2, with time varying weights. As time t increases and both firms are still alive, intensity λ_1 decreases at a geometric rate and converges to the smallest value z_1 in the heterogeneity distribution. Similarly, intensity $\gamma_1(t, t-k)$ is decreasing in time t (for given date of the default of firm 2) to the same limiting value z_1, but at a lower decay rate. The term structures of firm 1 are given by:

$$\begin{aligned} B_1(t,h) &= e^{-z_1 h}\frac{1+\frac{1-\pi}{\pi}e^{-(2t+h)\Delta z}}{1+\frac{1-\pi}{\pi}e^{-2t\Delta z}}, \quad \text{when both firms are still alive,} \\ B_1(t,h,k) &= e^{-z_1 h}\frac{1+\frac{1-\pi}{\pi}\frac{z_2}{z_1}e^{-(2t+h-k)\Delta z}}{1+\frac{1-\pi}{\pi}\frac{z_2}{z_1}e^{-(2t-k)\Delta z}}, \quad \text{when firm 2 defaulted at } t-k. \end{aligned}$$

These term structures are decreasing, with a long term spread equal to z_1 and independent of t. Moreover, as time t increases, the short term spreads $\lambda_1(t), \gamma_1(t, t-k)$ decrease, and the term structures become flatter, approaching the level z_1. These features of the term structures are explained by a greater concentration of the conditional heterogeneity distribution at the smallest value z_1 when time increases. For instance we get:

$$\begin{aligned} P\left[Z = z_1 \mid I_t\right] &= \frac{1}{1+\frac{1-\pi}{\pi}e^{-2t\Delta z}}, \text{ when both firms are still alive at } t, \\ &= \frac{1}{1+\frac{1-\pi}{\pi}\frac{z_2}{z_1}e^{-2t\Delta z}}, \text{ when firm 2 defaulted at } t-k, \end{aligned}$$

and this probability increases to one with time.

Example 5: Heterogeneity with gamma distribution

When the heterogeneity factor Z follows a gamma distribution with parameter ν:

$$\Psi\left(y\right) = \frac{1}{(1+y)^{\nu}},$$

the Archimedean copula characterizing the dependence between times to default reduces to a Clayton copula [see Clayton (1978), Oakes (1982)]. Note that the gamma distribution is continuous in $(0,\infty)$; in particular the lower bound of its support is zero. Thus there is a non-zero probability for the firms to be almost without default. The intensities are given by:

$$\begin{aligned} \lambda_1(t) &= \frac{\nu}{1+2t}, \\ \gamma_1(t,t-k) &= \frac{\nu+1}{1+2t-k}, \end{aligned}$$

and the term structures are:

$$\begin{aligned} r_1(t,h) &= -\frac{1}{h}\log B_1(t,h) = \frac{\nu}{h}\log\left(1+\frac{h}{1+2t}\right), \\ r_1(t,h,k) &= -\frac{1}{h}\log B_1(t,h,k) = \frac{\nu+1}{h}\log\left(1+\frac{h}{1+2t-k}\right). \end{aligned}$$

These term structures are decreasing, and converge to a zero long term spread. Moreover, as time t increases, the short term spreads decrease, and the term structures become flatter, converging to 0.

It is also interesting to discuss the discontinuity of the intensity of firm 1 when the second firm defaults.

Proposition 9 *The jump in the intensity is given by:*

$$\gamma_1(t,t^-) - \lambda_1(t) = -\frac{\psi^{''}(2t)}{\psi^{'}(2t)} = \frac{V\left[Z \mid Y_1 > t, Y_2 > t\right]}{E\left[Z \mid Y_1 > t, Y_2 > t\right]}.$$

Thus the jump is nonnegative; in particular it is zero if and only if the factor Z is constant, that is in the homogeneous case. This increase in the short rate spread of firm 1 when the second firm defaults corresponds to the positive default correlation induced by the common factor Z. Moreover, the size of the jump at time t is related to the dispersion of factor Z conditionally to $Y_1 > t, Y_2 > t$ [13] [14]. Finally note that the knowledge of the jump magnitude for any t is equivalent to the knowledge of Ψ' up to a multiplicative factor, that is to the knowledge of the copula.

Example 5 (continued): For gamma heterogeneity the amplitude of the jump in intensity is given by:

$$\gamma_1(t, t^-) - \lambda_1(t) = \frac{1}{1 + 2t},$$

and the relative amplitude is constant, equal to $1/\nu$. More generally, default of firm 2 has a multiplicative effect on the term structure of firm 1: $r_1(t, h, 0^+) = (1 + 1/\nu)\, r_1(t, h)$.

ii) First-to-default basket

The first-to-default term structure is given by:

$$C\,(t, h) = \frac{\Psi\,(2t + 2h)}{\Psi\,(2t)},$$

and is deduced from the term structure $B_1\,(t, h)$ of the underlying corporate bonds by a simple change of time unit: $h \to 2h$.

iii) Extension to an arbitrary number of firms

The basic model is easily extended to an arbitrary number of firms [see section 2.6]. The joint survivor function becomes:

$$\Psi\,(y_1, y_2, ..., y_N) = \Psi\,(y_1 + y_2 + ... + y_N)\,. \tag{29}$$

The expressions of prices of zero-coupon bonds are [see (20)]:

$$B_1(t, h, k_{m+1}, ..., k_N) = \frac{\Psi^{(N-m)}\,(Nt + h - k_{m+1} - ... - k_N)}{\Psi^{(N-m)}\,(Nt - k_{m+1} - ... - k_N)},$$

whereas the intensity is:

$$\gamma_1(t, t - k_{m+1}, ..., t - k_N) = -\frac{\Psi^{(N-m+1)}\,(Nt - k_{m+1} - ... - k_N)}{\Psi^{(N-m)}\,(Nt - k_{m+1} - ... - k_N)}.$$

[13] The dispersion of factor Z is also related to the strength of positive nonlinear dependence between times to default Y_1, Y_2 [see e.g. Gagliardini, Gouriéroux (2002)].

[14] Proposition 9 is also a consequence of the general result proved in Appendix 2 ii) and is deduced by taking $\underline{Z} = Z$, $\lambda_1^*(t) = \lambda_2^*(t) = Z$.

Thus the term structures depend on defaulted firms by their number $N-m$ and their average date of default $\overline{k}$, say:

$$\begin{aligned} B_1(t,h,k_{m+1},...,k_N) &= \frac{\Psi^{(N-m)}\left(Nt+h-(N-m)\,\overline{k}\right)}{\Psi^{(N-m)}\left(Nt-(N-m)\,\overline{k}\right)} \\ &= \frac{\Psi^{(N-m)}\left(\sum_{i=1}^{N}\min\left(Y_i,t\right)+h\right)}{\Psi^{(N-m)}\left(\sum_{i=1}^{N}\min\left(Y_i,t\right)\right)}, \\ \gamma_1(t,k_{m+1},...,k_N) &= -\frac{\Psi^{(N-m+1)}\left(Nt-(N-m)\,\overline{k}\right)}{\Psi^{(N-m)}\left(Nt-(N-m)\,\overline{k}\right)} \\ &= \frac{\Psi^{(N-m+1)}\left(\sum_{i=1}^{N}\min\left(Y_i,t\right)\right)}{\Psi^{(N-m)}\left(\sum_{i=1}^{N}\min\left(Y_i,t\right)\right)}. \end{aligned}$$

This possibility of aggregating the times to default is a direct consequence of the equidependence assumption.

Example 5 (continued): For an heterogeneity factor following a gamma distribution with parameter ν, the default intensity is given by:

$$\gamma_1(t,k_{m+1},...,k_N) = \frac{\nu+N-m+1}{Nt-(N-m)\,\overline{k}},$$

and the term structures are:

$$r_1(t,h,k_{m+1},...,k_N) = \frac{\nu+N-m}{h}\log\left(1+\frac{h}{Nt-(N-m)\,\overline{k}}\right).$$

These term structures are decreasing, and converge in the long run to zero. Note that such a situation is observed in practice. A typical example is default behaviour of firms with a low rating CCC, say, at a given date. The class CCC is often very heterogeneous including some good risks which have not been detected. After a large term h the firms from this class which are still alive correspond in fact to firms with a small default probability [see e.g. Carty (1997), Foulcher, Gouriéroux, Tiomo (2003)]. This change of default probability is due to the positive effect of the no default observed for the firm between t and $t+h$.

Furthermore, at each date of default of a firm, there is a multiplicative effect on the term structure, which increases at all terms, and afterwards converges to zero at a smaller rate. More precisely the jump of order $N-m$ has a relative effect on the intensity given by: $\left(\nu+N-m+1\right)^{-1}$. It depends on the number $N-m$ of defaulted firms only, not on times to default, and decreases with $N-m$. Note finally that this jump in intensities arise for all remaining firms simultaneously. The positive default correlation implies a correlation between the jumps in intensity.

iv) jth-to-default basket

It is also interesting to extend the result on credit derivatives to first-, second-, third-to-default baskets. Indeed let us denote $Y_{(1)} < Y_{(2)} < ... < Y_{(N)}$ the times to default ranked in increasing order and $D_1 = Y_{(1)}, D_2 = Y_{(2)} - Y_{(1)}, ..., D_N = Y_{(N)} - Y_{(N-1)}$ the interdefault durations. $D_1, D_2, ..., D_N$ are independent conditional on Z, with exponential distributions $\gamma(1, NZ)$, $\gamma(1, (N-1)Z)$, ..., $\gamma(1, Z)$, respectively. In particular their conditional survivor function is:

$$P[D_1 > d_1, ..., D_N > d_N \mid Z] = \exp -Z[Nd_1 + (N-1)d_2 + ... + d_N].$$

At date $t = 0$ the joint survivor function of $D_1, ..., D_N$ is:

$$S_d(d_1, ..., d_N) = \Psi[Nd_1 + (N-1)d_2 + ... + d_N]. \tag{30}$$

Example 6: Second-to-default basket

A second-to-default basket with residual maturity h pays 1\$ if the second default occurs after $t + h$. Its price at time t is given by:

$$C_2(t, h) = P\left[Y_{(2)} > t + h \mid I_t\right].$$

Different cases can be distinguished according to default histories of the firms at time t.
i) The price is zero if the second default occurred before t.
ii) If only the first default occurred before t, at time $t - k$, say, the price is given by:

$$\begin{aligned} C_2(t, h) &= P\left[Y_{(2)} > t + h \mid Y_{(1)} = t - k\right] \\ &= P[D_2 > k + h \mid D_1 = t - k] \\ &= \frac{\frac{\partial S_d}{\partial d_1}(t - k, k + h, 0, ..., 0)}{\frac{\partial S_d}{\partial d_1}(t - k, 0, 0, ..., 0)} \\ &= \frac{\Psi'[N(t-k) + (N-1)(k+h)]}{\Psi'[N(t-k)]}. \end{aligned}$$

iii) Finally, if no default occurred before date t, the price is given by:

$$\begin{aligned} C_2(t, h) &= P\left[Y_{(2)} > t + h \mid Y_{(1)} > t\right] \\ &= \frac{P[D_1 > t, D_1 + D_2 > t + h]}{P[D_1 > t]} \\ &= \frac{P[D_1 > t + h]}{P[D_1 > t]} + \frac{P[t < D_1 < t + h, D_1 + D_2 > t + h]}{P[D_1 > t]}. \end{aligned}$$

By using the conditional independence of D_1, D_2 given Z we have [see Appendix 6]:

$$P[t < D_1 < t + h, D_1 + D_2 > t + h] = N\{\Psi[(N-1)(t+h) + t] - \Psi[N(t+h)]\}.$$

Thus:

$$C_2(t,h) = \frac{N\Psi[(N-1)(t+h)+t] - (N-1)\Psi[N(t+h)]}{\Psi(Nt)}, \tag{31}$$

when no default occurs before t.

3.1.2 Models with common and idiosyncratic unobservable risk factors

The results above can be directly extended to distinguish between common and idiosyncratic factors.

i) The factor model

Let $Z, Z_1, ..., Z_N$ denote $N+1$ mutually independent factors. Let us assume that, conditionally to factors $Z, Z_1, ..., Z_N$, the times to default Y_i, $i = 1, ..., N$, are independent, and follow exponential distributions with parameters λ_i, $i = 1, ..., N$, given by:

$$\lambda_i = Z + Z_i, \qquad i = 1, ..., N.$$

The factors Z and $Z_1, ..., Z_N$ are interpreted as common and firm specific factors, respectively, which are constant through time, and affect default intensities of the firms. Default correlation is originated from the common factor Z.

Let us denote by $\Psi_c = \exp(-\psi_c)$ and $\Psi = \exp(-\psi)$ the real Laplace transforms of factors Z and Z_i, $i = 1, ..., N$, respectively. The joint survivor function of times to default $Y_1, ..., Y_N$ becomes:

$$\begin{aligned} S(y_1, ..., y_N) &= E\exp[-(Z+Z_1)y_1 - ... - (Z+Z_N)y_N] \\ &= E\exp[-Z(y_1 + ... + y_N)]\prod_{i=1}^{N} E\exp(-Z_i y_i) \\ &= \Psi_c(y_1 + ... + y_N)\prod_{i=1}^{N}\Psi(y_i). \end{aligned} \tag{32}$$

The nonlinear dependence can be summarized by the associated N-variate survivor copula. The times to default admit identical marginal distributions with survivor functions: $S_i(y_i) = \Psi_c(y_i)\Psi(y_i)$. Thus the survivor copula is:

$$C(u_1, ..., u_N) = \Psi_c\left[\sum_{i=1}^{N}(\Psi_c\Psi)^{-1}(u_i)\right]\prod_{i=1}^{N}\Psi\left[(\Psi_c\Psi)^{-1}(u_i)\right], \tag{33}$$

and provides a natural extension of the Archimedean copula of Section 3.1.1.

ii) The term structures

Let us derive the term structures for $N = 3$ firms. From (32) we deduce [see Appendix 5]:

$$\begin{aligned}
B_1(t,h) &= \frac{S(t+h,t,t)}{S(t,t,t)} = \frac{\Psi_c(3t+h)\,\Psi(t+h)}{\Psi_c(3t)\,\Psi(t)}, \\
B_1(t,h,k_3) &= \frac{\Psi_c(3t+h-k_3)\,\Psi(t+h)}{\Psi_c(3t-k_3)\,\Psi(t)} \frac{\psi_c^{'}(3t+h-k_3)+\psi^{'}(t-k_3)}{\psi_c^{'}(3t-k_3)+\psi^{'}(t-k_3)}, \\
B_1(t,h,k_2,k_3) &= \frac{\Psi_c(3t+h-k_2-k_3)\,\Psi(t+h)}{\Psi_c(3t-k_2-k_3)\,\Psi(t)} \\
&\cdot\Big\{\psi_c^{''}(3t+h-k_2-k_3)-\psi^{'}(t-k_2)\,\psi^{'}(t-k_3) \\
&-\psi_c^{'}(3t+h-k_3)\Big[\psi_c^{'}(3t+h-k_3)-\psi^{'}(t-k_2)-\psi^{'}(t-k_3)\Big]\Big\} \\
&\cdot\Big\{\psi_c^{''}(3t-k_2-k_3)-\psi^{'}(t-k_2)\,\psi^{'}(t-k_3) \\
&-\psi_c^{'}(3t-k_3)\Big[\psi_c^{'}(3t-k_3)-\psi^{'}(t-k_2)-\psi^{'}(t-k_3)\Big]\Big\}^{-1}.
\end{aligned}$$

The associated intensities are:

$$\begin{aligned}
\lambda_1(t) &= -\left.\frac{\partial B_1(t,h)}{\partial h}\right|_{h=0} = \psi_c^{'}(3t)+\psi^{'}(t), \\
\lambda_1(t,t-k_3) &= -\left.\frac{\partial B_1(t,h,k_3)}{\partial h}\right|_{h=0} = \psi_c^{'}(3t-k_3)+\psi^{'}(t)-\frac{\psi_c^{''}(3t-k_3)}{\psi_c^{'}(3t-k_3)+\psi^{'}(t-k_3)}, \\
\lambda_1(t,t-k_2,t-k_3) &= -\left.\frac{\partial B_1(t,h,k_2,k_3)}{\partial h}\right|_{h=0} = \psi_c^{'}(3t-k_2-k_3)+\psi^{'}(t) \\
&-\Big\{\psi_c^{''}(3t-k_2-k_3)-\psi^{'}(t-k_2)\,\psi^{'}(t-k_3) \\
&-\psi_c^{'}(3t-k_3)\Big[\psi_c^{'}(3t-k_3)-\psi^{'}(t-k_2)-\psi^{'}(t-k_3)\Big]\Big\}^{-1} \\
&\cdot\Big\{\psi_c^{'''}(3t-k_2-k_3) \\
&-\psi_c^{''}(3t-k_3)\Big[2\psi_c^{'}(3t-k_3)-\psi^{'}(t-k_2)-\psi^{'}(t-k_3)\Big]\Big\}
\end{aligned}$$

As expected the intensity jump involves the common component Ψ_c, but not the idiosyncratic component $\psi^{'}$. Contrary to Section 3.1.1 iii) the effect of previous default can no longer be summarized by the sum $k_2 + k_3$.

iii) First-to-default

The first-to-default term structure is given by:

$$C_N(t,h) = \frac{\Psi_c(Nt+Nh)\,\Psi(t+h)^N}{\Psi_c(Nt)\,\Psi(t)^N}. \tag{34}$$

The associated term structure of interest rates is:

$$r_{C,N}\left(t,h\right)=-\frac{1}{h}\log\frac{\Psi_{c}\left(Nt+Nh\right)}{\Psi_{c}\left(Nt\right)}-\frac{N}{h}\log\frac{\Psi\left(t+h\right)}{\Psi\left(t\right)}, \tag{35}$$

whereas the first-to-default intensity is:

$$r_{C,N}\left(t\right)=N\left[\psi_{c}^{'}\left(Nt\right)+\psi^{'}\left(t\right)\right]. \tag{36}$$

The latter formula illustrates the effect of the portfolio size. If the times to default are independent $r_{C,N}\left(t\right)=N\psi^{'}\left(t\right)$, and the intensity is a linear function of the size. Otherwise the default correlation effect is not negligible w.r.t. the idiosyncratic effect.

Example 4 (continued): When the common heterogeneity factor takes two values $z_1 < z_2$, we get:

$$\lim_{N\to\infty}\psi_{c}^{'}\left(Nt\right)=z_{1},$$

and for large portfolio size:

$$r_{C,N}\left(t\right)\simeq N\left[z_{1}+\psi^{'}\left(t\right)\right].$$

Example 5 (continued): For a gamma common heterogeneity factor: $\Psi_c(y) = 1/\left(1+y\right)^{\nu}$, the intensity becomes:

$$r_{C,N}\left(t\right)=\frac{\nu N}{1+Nt}+N\psi^{'}(t);$$

the effect of default correlation vanishes for large size portfolios.

3.2 Dynamic factor models and information sets

The factor models can easily be extended to include time varying factors. The aim of this section is to point out the importance of the information set, already mentioned in the theoretical literature [see e.g. Elliot, Jeanblanc, Yor (2000), Rutkowski (1999), Schonbucher, Schubert (2001)]. Indeed the intensities, intensity jumps, term structures and default correlations depend heavily on this set. The time varying factor is denoted by Z_t, $t \in \mathbb{R}^+$. The information set including the current and lagged factor values at time t is denoted by $\underline{Z_t}$. In particular $\underline{Z} = \underline{Z_\infty}$ is generated by the past, current and future values of the factor.

3.2.1 Complete information on the factor process and default history.

If the factor trajectory is entirely known by the investors, the results of Section 2 can be applied conditionally to $\underline{Z}$. With clear notation, we get:

$$
\begin{aligned}
B_1^*(t,h) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 > t, \underline{Z}\right] = \frac{S\left(t+h, t \mid \underline{Z}\right)}{S\left(t, t \mid \underline{Z}\right)}, \\
B_1^*(t,h,k) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 = t-k, \underline{Z}\right] = \frac{\frac{\partial S}{\partial y_2}\left(t+h, t-k \mid \underline{Z}\right)}{\frac{\partial S}{\partial y_2}\left(t, t-k \mid \underline{Z}\right)}, \\
r_1^*(t,h) &= -\frac{1}{h}\log B_1^*(t,h), \qquad r_1^*(t,h,k) = -\frac{1}{h}\log B_1^*(t,h,k), \\
\lambda_1^*(t) &= -\left.\frac{\partial B_1^*(t,h)}{\partial h}\right|_{h=0}, \qquad \gamma_1^*(t,t-k) = -\left.\frac{\partial B_1^*(t,h,k)}{\partial h}\right|_{h=0}.
\end{aligned}
$$

There is a jump in the intensities (conditionally to $\underline{Z}$), if and only if the infinitesimal default occurrences are correlated (conditionally to $\underline{Z}$):

$$
\begin{aligned}
& \gamma_1^*(t,t^-) - \lambda_1^*(t) \neq 0 \\
\iff & \lim_{dt\to 0} \frac{1}{dt^2} Cov\left[\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t, Y_2 > t, \underline{Z}\right] \neq 0.
\end{aligned}
$$

Thus both intensities and default correlations are computed from the same information set $\underline{Z}$.

Example 6: Let us consider the static factor model introduced in Section 3.1.1. The factor $Z_t = Z$ is time independent and $\underline{Z} = Z$. Conditionally to Z, the times to default are independent. We get no default correlation, whereas $\lambda_1^*(t) = \gamma_1^*(t,t-k) = Z$, $\forall t,k$, which implies no jump in intensities.

3.2.2 Information on default history only.

This framework has been considered in Section 2. We get:

$$
\begin{aligned}
B_1(t,h) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 > t,\right] = \frac{S\left(t+h, t\right)}{S\left(t, t\right)}, \\
B_1(t,h,k) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 = t-k\right] = \frac{\frac{\partial S}{\partial y_2}\left(t+h, t-k\right)}{\frac{\partial S}{\partial y_2}\left(t, t-k\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
S\left(y_1, y_2\right) &= P\left[Y_1 > y_1, Y_2 > y_2\right] \\
&= EP\left[Y_1 > y_1, Y_2 > y_2 \mid \underline{Z}\right] \\
&= E\, S\left(y_1, y_2 \mid \underline{Z}\right).
\end{aligned}
$$

Therefore the term structure $B_1(t,h)$ can be easily related to the term structure $B_1^*(t,h)$. We get:

$$\begin{aligned} B_1(t,h) &= \frac{S(t+h,t)}{S(t,t)} = \frac{E\ S(t+h,t \mid \underline{Z})}{E\ S(t,t \mid \underline{Z})} \\ &= E\left[B_1^*(t,h)\frac{S(t,t \mid \underline{Z})}{E\ S(t,t \mid \underline{Z})}\right]. \end{aligned} \tag{37}$$

Thus the term structure $B_1(t,h)$ corresponding to the smallest information is deduced from the term structure $B_1^*(t,h)$ corresponding to the largest information by averaging with respect to a modified probability for Z. The change of probability is $S(t,t \mid \underline{Z})/E[S(t,t \mid \underline{Z})]$.

Similarly the term structure $B_1(t,h,k)$ after default of firm 2 is also deduced by averaging $B_1^*(t,h,k)$ with a modified probability. But the change of probability is now: $\frac{\partial S}{\partial y_2}(t,t-k \mid \underline{Z})/E\left[\frac{\partial S}{\partial y_2}(t,t-k \mid \underline{Z})\right]$. The difference in the two changes of probability is due to the difference in the information sets.

The intensities $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ are computed as in Section 2, and there is a jump in the intensities if and only if the infinitesimal default occurrences are correlated conditional to the default history:

$$\begin{aligned} & \gamma_1(t,t^-) - \lambda_1(t) \neq 0 \\ \Longleftrightarrow\quad & \lim_{dt\to 0}\frac{1}{dt^2}Cov\left[\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1>t, Y_2>t\right] \neq 0. \end{aligned}$$

Example 6 (continued): When the static factor is integrated out we have noted in Section 3.1.1 that there is a jump in the intensities, whenever Z is not constant. Thus no jump and no default correlation exist when Z is observed, whereas jump in intensities and default correlation are spuriously created when the information diminishes and reduces to default history.

Example 7: In the firm value approach [Merton (1974)], two latent processes are introduced Z_t^1, Z_t^2, say, and the times to default are defined by:

$$Y_1 = \inf\left\{t : Z_t^1 < 0\right\}, \qquad Y_2 = \inf\left\{t : Z_t^2 < 0\right\}.$$

Z_t^i, $i = 1,2$, is usually interpreted as the difference between firm's asset values and liabilities.

i) If the trajectories of $\left(Z_t^1\right), \left(Z_t^2\right)$ are known, the times to default become deterministic. The intensities can be infinite and there is no default correlation (conditionally to $\underline{Z}$).

ii) Without the observations of firms's assets and liabilities, the default of the firm appears as imperfectly expected news, which creates the jump in intensities and the impression of default correlation. This discussion shows that the usual distinction done in the literature between structural and intensity (or reduced form) models is rather misleading. Indeed any (multivariate) duration model can be characterized by means of intensity functions (possibly infinite) and is automatically an intensity model. In fact the main difference is the information set, which is generally larger, including latent quantitative processes, in the so-called structural models.

Another remark is also important to understand the effect of information on jump intensities and default correlation. It is known by covariance analysis equation that:

$$
\begin{aligned}
& Cov\left[\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t, Y_2 > t\right] \\
= \; & Cov\left(E\left[\mathbb{I}_{t<Y_1<t+dt} \mid Y_1 > t, Y_2 > t, \underline{Z}\right], E\left[\mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t, Y_2 > t, \underline{Z}\right] \mid Y_1 > t, Y_2 > t\right) \\
& + E\left(Cov\left[\mathbb{I}_{t<Y_1<t+dt}, \mathbb{I}_{t<Y_2<t+dt} \mid Y_1 > t, Y_2 > t, \underline{Z}\right] \mid Y_1 > t, Y_2 > t\right).
\end{aligned}
$$

Thus the sign of default correlation can be completely modified by the choice of the information set. Examples 6 and 7 are special cases in which the second component of the RHS is equal to zero. The absence of default correlation at the informed level does not imply the absence of default correlation at the less informed level due to the first component in the RHS.

A similar remark can be done on the jump in intensities. Indeed we have:

$$
\begin{aligned}
& \gamma_1(t, t^-) - \lambda_1(t) \\
= \; & E\left[\gamma_1^*(t, t^-) \frac{\frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})}{E \frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})}\right] - E\left[\lambda_1^*(t) \frac{S\,(t, t \mid \underline{Z})}{E\; S\,(t, t \mid \underline{Z})}\right] \\
= \; & E\left(\left[\gamma_1^*(t, t^-) - \lambda_1^*(t)\right] \frac{S\,(t, t \mid \underline{Z})}{E\; S\,(t, t \mid \underline{Z})}\right) \\
& + E\left[\gamma_1^*(t, t^-) \left(\frac{\frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})}{E \frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})} - \frac{S\,(t, t \mid \underline{Z})}{E\; S\,(t, t \mid \underline{Z})}\right)\right] \\
= \; & E\left(\left[\gamma_1^*(t, t^-) - \lambda_1^*(t)\right] \frac{S\,(t, t \mid \underline{Z})}{E\; S\,(t, t \mid \underline{Z})}\right) \\
& + Cov\left[\gamma_1^*(t, t^-), \left(\frac{\frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})}{E \frac{\partial S}{\partial y_2}(t, t \mid \underline{Z})} - \frac{S\,(t, t \mid \underline{Z})}{E\; S\,(t, t \mid \underline{Z})}\right)\right],
\end{aligned} \tag{38}
$$

since the two probability changes have the same unitary mean. In Example 6 and 7, the first component of the RHS is zero, but the second component does not vanish. In fact we have to take into account the different probability changes involved in the expression of the intensities.

3.2.3 Information on default and factor history

Let us finally consider the intermediate case where the information includes default history and $\underline{Z_t}$. The same arguments as above will apply. We get:

$$\begin{aligned}
B_1(t,h,\underline{Z_t}) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 > t, \underline{Z_t}\right], \\
\lambda_1(t,\underline{Z_t}) &= \lim_{dt \to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 > t, \underline{Z_t}\right], \\
\gamma_1(t,t-k,\underline{Z_t}) &= \lim_{dt \to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 = t-k, \underline{Z_t}\right],
\end{aligned}$$

and so on, where the information introduced in the different expressions corresponds to date t. The remarks on jump intensities and default correlation of Section 2.2 remain valid after conditioning on $\underline{Z_t}$. Moreover it is easy to characterize the jump in intensities in terms of default correlation [see Appendix 2 ii)].

Proposition 10 *If the default risks are diversifiable (that is Y_1 and Y_2 are independent conditional on $\underline{Z}$)* [15]*:*

$$\gamma_1(t,t^-,\underline{Z_t}) - \lambda_1(t,\underline{Z_t}) = \lim_{dt \to 0} \frac{Cov\left[\lambda_1^*(t), \lambda_2^*(t-dt) \mid Y_1 > t, Y_2 > t-dt, \underline{Z_t}\right]}{E\left[\lambda_2^*(t-dt) \mid Y_1 > t, Y_2 > t-dt, \underline{Z_t}\right]}.$$

However some other results of Section 2 are no longer valid for dynamic factors. This is typically the case of Proposition 3 and its associated Corollaries. For instance the term structures of zero-coupon bonds computed when both firms are still alive no longer provide the same information as the short term spreads. To illustrate this point, let us note that:

$$\begin{aligned}
B_1(t,h,\underline{Z_t}) &= E\left[B_1^*(t,h) \mid Y_1 > t, Y_2 > t, \underline{Z_t}\right], \\
\lambda_1(t,\underline{Z_t}) &= E\left[\lambda_1^*(t) \mid Y_1 > t, Y_2 > t, \underline{Z_t}\right], \\
B_1(t,h,k,\underline{Z_t}) &= E\left[B_1^*(t,h,k) \mid Y_1 > t, Y_2 = t-k, \underline{Z_t}\right], \\
\gamma_1(t,t^-,\underline{Z_t}) &= E\left[\gamma_1^*(t,t^-) \mid Y_1 > t, Y_2 = t, \underline{Z_t}\right].
\end{aligned}$$

The dynamic factor models introduced in the literature generally satisfy the following assumption [see e.g. Lando (1998), Duffie, Singleton (1999), Duffie, Garleanu (2001), Jarrow, Lando, Yu (2001), Gouriéroux, Monfort, Polimenis (2003)].

Assumption A.2: For any t, $(Y_1 > t, Y_2 > t)$ is independent of $\underline{Z}$ conditional on $\underline{Z_t}$: $(Y_1 > t, Y_2 > t) \perp \underline{Z} \mid \underline{Z_t}$.

[15] The diversifiability assumption is satisfied by several models proposed in the literature, see e.g. Lando (1998), Duffie, Singleton (1999), Duffie, Garleanu (2001), Jarrow, Lando, Yu (2001), Gouriéroux, Monfort, Polimenis (2003).

Assumption A.2 is equivalent to:

$$
\begin{aligned}
& P\left[Y_1 > t, Y_2 > t \mid \underline{Z}\right] = P\left[Y_1 > t, Y_2 > t \mid \underline{Z_t}\right] \\
\Longleftrightarrow \quad & \exp\left[-\int_0^t \lambda_1^*(s)ds - \int_0^t \lambda_2^*(s)ds\right] = E\left(\exp\left[-\int_0^t \lambda_1^*(s)ds - \int_0^t \lambda_2^*(s)ds\right] \mid \underline{Z_t}\right).
\end{aligned}
$$

It is satisfied if $\lambda_1^*(t)$ and $\lambda_2^*(t)$ are functions of $\underline{Z_t}$. Under Assumption A.2, we get from Corollary 6:

$$
\begin{aligned}
B_1(t,h,\underline{Z_t}) &= E\left[B_1^*(t,h) \mid \underline{Z_t}\right] \\
&= E\Big[e^{-[\Lambda_1^*(t+h)-\Lambda_1^*(t)]-[\Lambda_2^*(t+h)-\Lambda_2^*(t)]} \\
&\quad + \int_t^{t+h} \lambda_2^*(y)\, e^{-[\Lambda_1^*(y)-\Lambda_1^*(t)]-[\Lambda_1^*(y)-\Lambda_1^*(t)]-\Gamma_1^*(t+h-y,y)} dy \mid \underline{Z_t}\Big],
\end{aligned}
$$

which does not coincide with the expression of Corollary 6 after replacing the intensities λ, γ by $\lambda\left(.;\underline{Z_t}\right)$, $\gamma\left(.;\underline{Z_t}\right)$.

Example 8: The model with common and idiosyncratic unobservable risk factors can be directly extended to the dynamic framework. Let us introduce $[Z_1(t)]$, $[Z_2(t)]$, $[Z(t)]$, three independent factor processes, and assume that Y_1, Y_2 are independent conditionally to $\underline{Z_1}, \underline{Z_2}, \underline{Z}$ with conditional intensities:

$$\lambda_i^*(t) = Z_i(t) + Z(t), \quad i = 1,2.$$

The term structure of zero-coupon prices can be computed with different information sets.
i) With complete information on factor processes, we get:

$$
\begin{aligned}
B_1^*(t,h) &= B_1^*(t,h,k) = \exp\left[-\int_t^{t+h} Z_1(s)ds - \int_t^{t+h} Z(s)ds\right], \\
\lambda_1^*(t) &= \gamma_1^*(t,t-k) = Z_1(t) + Z(t).
\end{aligned}
$$

ii) With partial information on all factor processes, we get:

$$
\begin{aligned}
B_1(t,h,\underline{Z_{1t}},\underline{Z_{2t}},\underline{Z_t}) &= B_1(t,h,k,\underline{Z_{1t}},\underline{Z_{2t}},\underline{Z_t}) \\
&= E\left[\exp - \int_t^{t+h} Z_1(s)ds \mid \underline{Z_{1t}}\right] E\left[\exp - \int_t^{t+h} Z(s)ds \mid \underline{Z_t}\right], \\
\lambda_1(t,\underline{Z_{1t}},\underline{Z_{2t}},\underline{Z_t}) &= \gamma_1(t,t^-,\underline{Z_{1t}},\underline{Z_{2t}},\underline{Z_t}) = Z_1(t) + Z(t).
\end{aligned}
$$

The model admits no jump in intensity.
iii) With partial information on the common factor and no information on the

idiosyncratic factors, we get:

$$
\begin{aligned}
B_1(t,h,\underline{Z_t}) &= \frac{E\left[\exp - \int_0^{t+h} Z_1(s)ds\right]}{E\left[\exp - \int_0^t Z_1(s)ds\right]} E\left[\exp - \int_t^{t+h} Z(s)ds \mid \underline{Z_t}\right], \\
\lambda_1(t,\underline{Z_t}) &= E\left[Z_1(t)\frac{\exp - \int_0^t Z_1(s)ds}{E\exp - \int_0^t Z_1(s)ds}\right] + Z(t).
\end{aligned}
$$

The possible jump in intensities is given by [see Proposition 10]:

$$
\begin{aligned}
& \gamma_1(t,t^-,\underline{Z_t}) - \lambda_1(t,\underline{Z_t}) \\
= \; & \frac{Cov\left[Z_1(t)+Z(t), Z_2(t)+Z(t) \mid Y_1>t, Y_2>t, \underline{Z_t}\right]}{E\left[Z_2(t)+Z(t) \mid Y_1>t, Y_2>t, \underline{Z_t}\right]} \\
= \; & \frac{Cov\left[Z_1(t), Z_2(t) \mid Y_1>t, Y_2>t, \underline{Z_t}\right]}{E\left[Z_2(t) \mid Y_1>t, Y_2>t, \underline{Z_t}\right] + Z(t)}.
\end{aligned}
$$

It is easily checked that the conditional distribution of $[Z_1(t), Z_2(t)]$ given $Y_1 > t, Y_2 > t, \underline{Z_t}$ is deduced from the risk neutral distribution of $Z_1(t), Z_2(t)$ by the change of density:

$$
\frac{\exp - \int_0^t Z_1(s)ds}{E\exp - \int_0^t Z_1(s)ds} \frac{\exp - \int_0^t Z_2(s)ds}{E\exp - \int_0^t Z_2(s)ds}.
$$

Therefore $Z_1(t)$ and $Z_2(t)$ are also independent conditionally to $Y_1 > t, Y_2 > t, \underline{Z_t}$, and there is no jump in the intensities. This independence is a consequence of the additive decomposition of $\lambda_1^*(t)$ as $Z_1(t) + Z(t)$. The independence is no longer satisfied if $\lambda_1^*(t) = a(Z_1(t), Z(t))$ involves cross effects of common and idiosyncratic factors.
iv) With the information on default histories only, we get:

$$
\begin{aligned}
B_1(t,h) &= \frac{E\left[\exp - \int_0^{t+h} Z_1(s)ds\right]}{E\left[\exp - \int_0^t Z_1(s)ds\right]} \frac{E\left[\exp -2\int_0^t Z(s)ds - \int_t^{t+h} Z(s)ds\right]}{E\left[\exp -2\int_0^t Z(s)ds\right]}, \\
\lambda_1(t) &= E\left[Z_1(t)\frac{\exp - \int_0^t Z_1(s)ds}{E\exp - \int_0^t Z_1(s)ds}\right] + E\left[Z(t)\frac{\exp -2\int_0^t Z(s)ds}{E\exp -2\int_0^t Z(s)ds}\right].
\end{aligned}
$$

In this case there is a jump in intensities:

$$
\gamma_1(t,t^-) - \lambda_1(t) = \frac{\underset{Q}{V}[Z(t)]}{\underset{Q_2}{E}[Z_2(t)] + \underset{Q}{E}[Z(t)]},
$$

where Q is deduced from the risk neutral distribution of $Z(t)$ by the change of density:

$$
\frac{\exp -2\int_0^t Z(s)ds}{E\exp -2\int_0^t Z(s)ds},
$$

and Q_2 is deduced from the risk neutral distribution of $Z_2(t)$ by the change of density:

$$\frac{\exp - \int_0^t Z_2(s)ds}{E \exp - \int_0^t Z_2(s)ds}.$$

Note that Assumption A.2 is not satisfied in the last two cases iii) and iv).

It is important to realize that the choice of the relevant information set is not completely unambiguous. Indeed, even if the factors cannot be observed directly, in affine models for instance they can be recovered from corporate bond prices [see e.g. Duffie, Kan (1996), and Gouriéroux, Monfort, Polimenis (2003)], with clear implications for the analysis of default correlation.

4 Concluding remarks

In this paper we provide different characterizations for the joint distribution of corporate times to default with general dependence structure. We analyse the implications of default correlation on the patterns of the term structures of corporate bonds and credit derivatives.

A key feature is that the spread term structure of a firm generally admits discontinuities at default dates of other firms. Similarly, the default intensity of a firm features a jump at default dates of other firms. We discuss carefully the links between intensity jumps and default correlation. Intensity jumps provide an alternative measure of default correlation, with a clear financial interpretation.

Finally we emphasize the importance of the information set for the discussion of default correlation. Indeed the sign and size of default correlation (or intensity jumps) heavily depend on the selected information set. We point out that the choice of the relevant information set is not completely unambiguous when the unobservable factors can be recovered from corporate bond prices.

Appendix 1
Term structure of corporate bonds when one firm defaulted earlier

Let us consider the price at time t of the zero-coupon bonds issued by firm 1. If firm 2 defaulted at the previous date $t-k$, the price at time t of this bond with residual maturity h is given by:

$$\begin{aligned} B_1(t,h,k) &= P\left[Y_1 > t+h \mid Y_1 > t, Y_2 = t-k\right] \\ &= \frac{P\left[Y_1 > t+h \mid Y_2 = t-k\right]}{P\left[Y_1 > t \mid Y_2 = t-k\right]} \\ &= \frac{\int_{t+h}^{\infty} f(y_1 \mid t-k)dy_1}{\int_t^{\infty} f(y_1 \mid t-k)dy_1} = \frac{\int_{t+h}^{\infty} f(y_1, t-k)dy_1}{\int_t^{\infty} f(y_1, t-k)dy_1} \\ &= \frac{\frac{\partial S}{\partial y_2}(t+h, t-k)}{\frac{\partial S}{\partial y_2}(t, t-k)}, \end{aligned}$$

where $f(y_1 \mid y_2)$ and $f(y_1, y_2) = \partial^2 S(y_1, y_2)/\partial y_1 \partial y_2$ are the conditional density of Y_1 given Y_2, and the joint density of Y_1,Y_2, respectively.

Appendix 2
Default correlation and jumps in intensities

i) A general result

To prove the result, it is useful to introduce the associated counting processes N_1, N_2, where $N_j(t) = 0$, if $Y_j > t$, $N_j(t) = 1$, otherwise. Note that the process increment $dN_j(t)$ is dichotomous with admissible values $0, 1$. Let us rewrite the expression of intensities in terms of the counting processes. We get:

$$\begin{aligned}
\lambda_1\left(t, \underline{Z_t}\right) &= \lim_{dt\to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 > t, \underline{Z_t}\right] \\
&= \lim_{dt\to 0} \frac{1}{dt} E\left[dN_1(t) \mid N_1(t) = 0, N_2(t) = 0, \underline{Z_t}\right] \\
&= \lim_{dt\to 0} \frac{1}{dt} E\left[dN_1(t) \mid N_1(t) = 0, N_2(t^-) = 0, dN_2(t^-) = 0, \underline{Z_t}\right],
\end{aligned}$$

where $dN_2(t^-) = N_2(t) - N_2(t-dt)$, and (Z_t) is a factor. Similarly we have:

$$\begin{aligned}
&\gamma_1\left(t, t^-, \underline{Z_t}\right) \\
= &\lim_{k\to 0} \lim_{dt\to 0} \frac{1}{dt} P\left[Y_1 < t+dt \mid Y_1 > t, Y_2 = t-k, \underline{Z_t}\right] \\
= &\lim_{k\to 0} \lim_{dt\to 0} \frac{1}{dt} E\left[dN_1(t) \mid N_1(t) = 0, N_2((t-k)^-) = 0, dN_2((t-k)^-) = 1, \underline{Z_t}\right] \\
= &\lim_{dt\to 0} \frac{1}{dt} E\left[dN_1(t) \mid N_1(t) = 0, N_2(t^-) = 0, dN_2(t^-) = 1, \underline{Z_t}\right].
\end{aligned}$$

Therefore both intensities admit interpretations in terms of conditional expectations of $dN_1(t)$ on $dN_2(t^-)$, given $N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}$. Since $dN_1(t)$ and $dN_2(t^-)$ are dichotomous qualitative variables, the conditional expectation coincides with the linear regression. Thus we have:

$$\begin{aligned}
&E\left[dN_1(t) \mid N_1(t) = 0, N_2(t^-) = 0, dN_2(t^-), \underline{Z_t}\right] \\
= &E\left[dN_1(t) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right] \\
&+ \frac{Cov\left[dN_1(t), dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]}{V\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]} \\
&\cdot \left(dN_2(t^-) - E\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]\right).
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\gamma_1\left(t, t^-, \underline{Z_t}\right) - \lambda_1\left(t, \underline{Z_t}\right) &= \lim_{dt\to 0} \frac{1}{dt} \frac{Cov\left[dN_1(t), dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]}{V\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]} \\
&= \lim_{dt\to 0} \frac{Cov\left[dN_1(t), dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]/dt^2}{E\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]/dt},
\end{aligned}$$

since the expectation and the variance of $dN_2(t^-)$ are equivalent. For instance, when there is no factor, the numerator reduces to the expression in (10) in the text.

ii) Diversifiable risk

Let us now assume that the default risks are diversifiable, that is the processes N_1 and N_2 are independent conditionally to $\underline{Z}$. The intensities with full information on the factors are: $\lambda_j^*(t) = \lim_{dt \to 0} E\left[dN_j(t) \mid N_j(t) = 0, \underline{Z}\right]/dt,\ j = 1, 2$. By the covariance analysis equation:

$$
\begin{aligned}
& Cov\left[dN_1(t), dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]/dt^2 \\
= \; & Cov\left\{E\left[dN_1(t) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z}\right]/dt,\right. \\
& \left. E\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z}\right]/dt \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right\} \\
& +E\left\{Cov\left[dN_1(t), dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z}\right] \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right\}/dt^2 \\
\to \; & Cov\left[\lambda_1^*(t), \lambda_2^*\left(t^-\right) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right] \\
& \text{(by the diversifiability assumption).}
\end{aligned}
$$

Similarly we get:

$$
E\left[dN_2(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]/dt \to E\left[\lambda_2^*\left(t^-\right) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right].
$$

Thus the jump in intensities is given by:

$$
\gamma_1\left(t, t^-, \underline{Z_t}\right) - \lambda_1\left(t, \underline{Z_t}\right) = \frac{Cov\left[\lambda_1^*(t), \lambda_2^*(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]}{E\left[\lambda_2^*(t^-) \mid N_1(t) = 0, N_2(t^-) = 0, \underline{Z_t}\right]}.
$$

Appendix 3
Flat term structure

If the term structures are flat when both firms are still alive, the joint survivor function admits the representation:

$$S(y_1, y_2) = \exp\left[-\lambda_1(y_2)(y_1 - y_2) - \Lambda_1(y_2) - \Lambda_2(y_2)\right], \text{ for } y_1 \geq y_2,$$

and:

$$S(y_1, y_2) = \exp\left[-\lambda_2(y_1)(y_2 - y_1) - \Lambda_1(y_1) - \Lambda_2(y_1)\right], \text{ for } y_1 < y_2.$$

Let us first derive the conditions on functions λ_1, λ_2 such that S is a well-defined bivariate survivor function.

i) Conditions on λ_1, λ_2

The survivor function is well-defined and corresponds to a continuous distribution iff:

a) $S(y_1, 0)$ and $S(0, y_2)$ are univariate survivor functions;

b) the density associated to S is positive:

$$\frac{\partial^2 S}{\partial y_1 \partial y_2}(y_1, y_2) \geq 0, \ \forall y_1 \neq y_2,$$

that is the function is differentiable;

c) the probability mass on $\{(y_1, y_2) : y_1 = y_2\}$ is equal to zero.

Let us consider condition a). We get:

$$S(y_1, 0) = \exp\left[-\lambda_1(0) y_1\right],$$

which is the survivor function of an exponential distribution, if $\lambda_1(0) > 0$. Similarly, we get the necessary condition $\lambda_2(0) > 0$. In particular $S(0, 0) = 1$ and the total mass is equal to one.

Let us now consider condition b). For $y_1 > y_2$ we have:

$$\frac{\partial S}{\partial y_2}(y_1, y_2) = -S(y_1, y_2)\left[\frac{d\lambda_1}{dt}(y_2)(y_1 - y_2) + \lambda_2(y_2)\right], \quad \text{(a.1)}$$

$$\begin{aligned} \frac{\partial^2 S}{\partial y_1 \partial y_2}(y_1, y_2) &= S(y_1, y_2)\lambda_1(y_2)\left[\frac{d\lambda_1}{dt}(y_2)(y_1 - y_2) + \lambda_2(y_2)\right] - S(y_1, y_2)\frac{d\lambda_1}{dt}(y_2) \\ &= S(y_1, y_2)\left\{\frac{d\lambda_1}{dt}(y_2)\left[\lambda_1(y_2)(y_1 - y_2) - 1\right] + \lambda_1(y_2)\lambda_2(y_2)\right\}. \end{aligned}$$

Thus the nonnegativity condition of the second order cross derivative becomes:

$$\frac{d\lambda_1}{dt}(y_2)\left[\lambda_1(y_2)\left(y_1-y_2\right)-1\right]+\lambda_1(y_2)\lambda_2(y_2)\geq 0, \quad \forall y_1>y_2. \tag{a.2}$$

We see that necessarily $d\lambda_1/dt(y_2) \geq 0$ by letting $y_1 \to \infty$. Moreover, we have $d\lambda_1/dt(y_2) \leq \lambda_1(y_2)\lambda_2(y_2)$ by considering the limiting condition $y_1 \to y_2$. The two inequalities are also sufficient for (a.2). Thus condition (a.2) is equivalent to:

$$0 \leq \frac{d\lambda_1}{dt}(y_2) \leq \lambda_1(y_2)\lambda_2(y_2), \quad \forall y_2 \geq 0.$$

Similarly, by considering the symmetric case $y_1 < y_2$, we deduce the condition:

$$0 \leq \frac{d\lambda_2}{dt}(y_1) \leq \lambda_1(y_1)\lambda_2(y_1), \quad \forall y_1 \geq 0.$$

Finally, in order to verify that the distribution associated to S has no mass on the diagonal $\{y_1 = y_2\}$, let us prove that the integrals of the density $\partial^2 S/\partial y_1 \partial y_2$ on the two triangles $\{y_1 > y_2\}$ and $\{y_1 < y_2\}$ sum up to 1. Indeed, for the triangle $\{y_1 > y_2\}$ we get:

$$\begin{aligned} I_1 &= \int_0^\infty \int_{y_2}^\infty \frac{\partial^2 S}{\partial y_1 \partial y_2}\left(y_1,y_2\right) dy_1 dy_2 = \int_0^\infty \frac{\partial S}{\partial y_2}\left(y_1,y_2\right)\Big|_{y_1=y_2}^{y_1=\infty} dy_2 \\ &= \int_0^\infty \lambda_2(y_2)\exp\left[-\Lambda_1\left(y_2\right)-\Lambda_2\left(y_2\right)\right] dy_2. \end{aligned}$$

Similarly the integral of the density over the triangle $\{y_1 < y_2\}$ is given by:

$$I_2 = \int_0^\infty \lambda_1(y_1)\exp\left[-\Lambda_1\left(y_1\right)-\Lambda_2\left(y_1\right)\right] dy_1.$$

Therefore:

$$I_1 + I_2 = \int_0^\infty \left[\lambda_1(y)+\lambda_2(y)\right]\exp\left[-\Lambda_1\left(y\right)-\Lambda_2\left(y\right)\right] dy = 1.$$

ii) Intensities

Let us now derive the default intensity of firm 1 when the second firm has defaulted at $t-k$. From equations (9) and (a.1) we get:

$$\gamma_1\left(t,t-k\right) = -\frac{\partial}{\partial y_1}\left[\log -\frac{\partial S}{\partial y_2}\right]\left(t,t-k\right) = \lambda_1\left(t-k\right) - \frac{\lambda_1^{'}(t-k)}{\lambda_1^{'}(t-k)k+\lambda_2(t-k)}.$$

iii) Term structure

Finally, let us derive from Corollary 6 the term structure of firm 1 when the second firm has defaulted at $t-k$. We get:

$$\begin{aligned} B_1\left(t,h,k\right) &= e^{-\lambda_1(t-k)h}\exp\left[\int_t^{t+h}\frac{\lambda_1^{'}(t-k)}{\lambda_1^{'}(t-k)\left(s-t+k\right)+\lambda_2(t-k)}ds\right] \\ &= e^{-\lambda_1(t-k)h}\exp\left[\int_{\lambda_1^{'}(t-k)k}^{\lambda_1^{'}(t-k)(k+h)}\frac{1}{s+\lambda_2(t-k)}ds\right] \\ &= e^{-\lambda_1(t-k)h}\frac{\lambda_1^{'}(t-k)\left(k+h\right)+\lambda_2(t-k)}{\lambda_1^{'}(t-k)k+\lambda_2(t-k)} \\ &= e^{-\lambda_1(t-k)h}\left[1+\frac{\lambda_1^{'}(t-k)}{\lambda_1^{'}(t-k)k+\lambda_2(t-k)}h\right]. \end{aligned} \tag{a.3}$$

In particular if $\lambda_1^{'}(t-k)=0$, we get:

$$B_1\left(t,h,k\right)=\exp\left[-\int_t^{t+h}\gamma_1\left(s,t-k\right)ds\right]=e^{-\lambda_1(t-k)h}.$$

Thus the term structure of firm 1 is flat even after the default of firm 2, and it depends only on its date of occurrence.

Appendix 4
MMPH model

The aim of this Appendix is to prove Proposition 8. Before we need the following Lemma.

Lemma A.1: Let ψ be the log-Laplace transform of a positive variable Z: $\psi(y) = -\log E\left[\exp\left(-yZ\right)\right]$. Let $z_1 \geq 0$ be the smallest value in the support of Z. Then function:

$$y \longmapsto \frac{\psi(y)}{y},$$

is decreasing, and:

$$\lim_{y\to\infty} \frac{\psi(y)}{y} = z_1.$$

Proof: We have:

$$\begin{aligned}\frac{d}{dy}\frac{\psi(y)}{y} &= \frac{\psi^{'}(y)y - \psi(y)}{y^2} = -\frac{1}{y^2}\left[\underbrace{\psi\left(y\right) + \left(-y\right)\psi^{'}\left(y\right)}_{\geq \psi(0)=0}\right] \\ &\leq 0,\end{aligned}$$

since ψ is concave. Let us now compute the limit of $\psi\left(y\right)/y$ when $y \to \infty$. We have $Z \geq z_1$ with probability 1 [resp. $Z < z_1^*$ with probability $P\left(Z \leq z_1^*\right) > 0$, for any $z_1^* > z_1$]. We deduce that:

$$P\left(Z \leq z_1^*\right)\exp\left(-yz_1^*\right) \leq E\left[\exp\left(-yZ\right)\right] \leq \exp\left(-yz_1\right),$$

and:

$$z_1 \leq \lim_{y\to\infty}\inf \frac{\psi(y)}{y} \leq \lim_{y\to\infty}\sup\frac{\psi(y)}{y} \leq z_1^*, \ \ \forall z_1^* > z_1.$$

Q.E.D.

Let us now prove Proposition 8. Since:

$$\begin{aligned} r_1(t,h) &= -\frac{1}{h}\log E\left[\exp\left(-hZ\right) \mid Y_1 > t, Y_2 > t\right], \\ r_1(t,h,k) &= -\frac{1}{h}\log E\left[\exp\left(-hZ\right) \mid Y_1 > t, Y_2 = t-k\right],\end{aligned}$$

i) and ii) follow immediately from Lemma A.1 since the lowest point in the support of the distribution of Z given I_t is z_1. Let us now consider iii). From (22) we have:

$$\begin{aligned}\frac{\partial}{\partial t} r_1(t,h) &= \frac{1}{h}\frac{\partial}{\partial t}\left[\psi\left(2t+h\right) - \psi(2t)\right] \\ &= \frac{2}{h}\left[\psi^{'}\left(2t+h\right) - \psi^{'}\left(2t\right)\right] \leq 0,\end{aligned}$$

since ψ is concave.

Moreover:

$$\begin{aligned}
\frac{\partial}{\partial t}B_1(t,h,k) &= \frac{\partial}{\partial t}\frac{E\left[Ze^{-(2t+h-k)Z}\right]}{E\left[Ze^{-(2t-k)Z}\right]} \\
&= -2\frac{E\left[Z^2e^{-(2t+h-k)Z}\right]E\left[Ze^{-(2t-k)Z}\right]-E\left[Ze^{-(2t+h-k)Z}\right]E\left[Z^2e^{-(2t-k)Z}\right]}{E\left[Ze^{-(2t-k)Z}\right]^2} \\
&= -2\overset{\widetilde{Q}}{cov}\left[Z,\exp\left(-hZ\right)\right]\geq 0,
\end{aligned}$$

where $\widetilde{Q}$ is the distribution with density $ze^{-(2t-k)z}/E\left[Ze^{-(2t-k)Z}\right]G(dz)$, and G is the distribution of Z.

Finally from Lemma A.1:

$$B_1(t,h)=\frac{\Psi\left(2t+h\right)}{\Psi\left(2t\right)}\simeq\exp\left(-z_1h\right),$$

for $t\to\infty$, and similarly for $B_1(t,h,k)$.

Appendix 5
Model with idiosyncratic factors

The joint survivor function of Y_1, Y_2, Y_3 is given by:

$$S(y_1, y_2, y_3) = \Psi_c(y_1 + y_2 + y_3)\Psi(y_1)\Psi(y_2)\Psi(y_3).$$

Their derivatives with respect to y_3 and y_2, y_3 are given by:

$$\begin{aligned}
\frac{\partial S}{\partial y_3}(y_1, y_2, y_3) &= \Psi_c'(y_1 + y_2 + y_3)\Psi(y_1)\Psi(y_2)\Psi(y_3) \\
&\quad +\Psi_c(y_1 + y_2 + y_3)\Psi(y_1)\Psi(y_2)\Psi'(y_3) \\
&= S(y_1, y_2, y_3)\left[\frac{\Psi_c'(y_1 + y_2 + y_3)}{\Psi_c(y_1 + y_2 + y_3)} + \frac{\Psi'(y_3)}{\Psi(y_3)}\right] \\
&= -S(y_1, y_2, y_3)\left[\psi_c'(y_1 + y_2 + y_3) + \psi'(y_3)\right],
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial^2 S}{\partial y_3 \partial y_2}(y_1, y_2, y_3) &= \Psi_c''(y_1 + y_2 + y_3)\Psi(y_1)\Psi(y_2)\Psi(y_3) \\
&\quad +\Psi_c'(y_1 + y_2 + y_3)\Psi(y_1)\Psi'(y_2)\Psi(y_3) \\
&\quad +\Psi_c'(y_1 + y_2 + y_3)\Psi(y_1)\Psi(y_2)\Psi'(y_3) \\
&\quad +\Psi_c(y_1 + y_2 + y_3)\Psi(y_1)\Psi'(y_2)\Psi'(y_3) \\
&= S(y_1, y_2, y_3)\left[\frac{\Psi_c''(y_1 + y_2 + y_3)}{\Psi_c(y_1 + y_2 + y_3)}\right. \\
&\quad +\frac{\Psi_c'(y_1 + y_2 + y_3)}{\Psi_c(y_1 + y_2 + y_3)}\left(\frac{\Psi'(y_2)}{\Psi(y_2)} + \frac{\Psi'(y_3)}{\Psi(y_3)}\right) \\
&\quad \left.+\frac{\Psi'(y_2)}{\Psi(y_2)}\frac{\Psi'(y_3)}{\Psi(y_3)}\right] \\
&= -S(y_1, y_2, y_3)\left\{\psi_c''(y_1 + y_2 + y_3) - \psi_c'(y_1 + y_2 + y_3)^2\right. \\
&\quad \left.-\psi_c'(y_1 + y_2 + y_3)\left[\psi_c'(y_2) + \psi_c'(y_3)\right] - \psi_c'(y_2)\psi_c'(y_3)\right\}.
\end{aligned}$$

Thus the term structure is given by:

$$B_1(t, h) = \frac{S(t + h, t, t)}{S(t, t, t)} = \frac{\Psi_c(3t + h)\Psi(t + h)}{\Psi_c(3t)\Psi(t)},$$

$$\begin{aligned}
B_1(t, h, k_3) &= \frac{\frac{\partial S}{\partial y_3}(t + h, t, t - k_3)}{\frac{\partial S}{\partial y_3}S(t, t, t - k_3)} \\
&= \frac{\Psi_c(3t + h - k_3)\Psi(t + h)}{\Psi_c(3t - k_3)\Psi(t)}\frac{\psi_c'(3t + h - k_3) + \psi'(t - k_3)}{\psi_c'(3t - k_3) + \psi'(t - k_3)},
\end{aligned}$$

$$
\begin{aligned}
B_1\left(t,h,k_2,k_3\right) &= \frac{\frac{\partial^2 S}{\partial y_2 \partial y_3} S\left(t+h,t-k_2,t-k_3\right)}{\frac{\partial^2 S}{\partial y_2 \partial y_3} S\left(t,t-k_2,t-k_3\right)} \\
&= \frac{\Psi_c\left(3t+h-k_2-k_3\right)\Psi\left(t+h\right)}{\Psi_c\left(3t-k_2-k_3\right)\Psi\left(t\right)} \\
&\quad \cdot \frac{\begin{array}{c}\psi_c^{''}\left(3t+h-k_2-k_3\right)-\psi^{'}\left(t-k_2\right)\psi^{'}\left(t-k_3\right) \\ -\psi_c^{'}\left(3t+h-k_3\right)\left[\psi_c^{'}\left(3t+h-k_3\right)-\psi^{'}\left(t-k_2\right)-\psi^{'}\left(t-k_3\right)\right]\end{array}}{\begin{array}{c}\psi_c^{''}\left(3t-k_2-k_3\right)-\psi^{'}\left(t-k_2\right)\psi^{'}\left(t-k_3\right) \\ -\psi_c^{'}\left(3t-k_3\right)\left[\psi_c^{'}\left(3t-k_3\right)-\psi^{'}\left(t-k_2\right)-\psi^{'}\left(t-k_3\right)\right]\end{array}}.
\end{aligned}
$$

Appendix 6
Second-to-default in a MMPH model with N firms

We have:

$$\begin{aligned}
P\left[t < D_1 < t+h, D_1 + D_2 > t+h\right] &= E\left[\int_t^{t+h} e^{-(N-1)Z(t+h-y)} NZe^{-NZy} dy\right] \\
&= NE\left[e^{-(N-1)Z(t+h)} Z \int_t^{t+h} e^{-Zy} dy\right] \\
&= NE\left[e^{-(N-1)Z(t+h)} \left(e^{-Zt} - e^{-Z(t+h)}\right)\right] \\
&= NE\left[e^{-Z[(N-1)(t+h)+t]} - e^{-NZ(t+h)}\right] \\
&= N\left(\Psi\left[(N-1)(t+h)+t\right] - \Psi\left[N(t+h)\right]\right).
\end{aligned}$$

REFERENCES

Bansal, R., and H., Zhou, (2002): "Term Structure of Interest Rates with Regime Shifts", Journal of Finance, 57, 1997-2043.

Bielecki, T., and M., Rutkowski (2002): *Credit Risk: Modeling, Valuation and Hedging,* Springer.

Bremaud, P., (1981): *Point Processes and Queues. Martingale Dynamics,* Springer Series in Statistics, New-York.

Carty, L., (1997): "Moody's Rating Migration and Credit Quality Correlation 1920-1996: Special Comment", Moody's Investor Service, New-York.

Chen, R., and J., Huang, (2001): "Credit Spread Bounds and Their Implication for Credit Risk Modelling", Rutgers University DP.

Clayton, D., (1978): "A Model for Association in Bivariate Life-Tables and its Application in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence", Biometrika, 65, 141-151.

Cox, D., (1972): "Regression Models and Life Tables", JRSS B, 34, 187-220.

Cox, D., and P., Lewis, (1972): "Multivariate Point Processes", Proc. 6^{th} Berk. Symposium, 3, 401-448.

Cox, D., and D., Oakes, (1984): *Analysis of Survival Data*, Chapman & Hall.

David, A., (2002): "The Dynamic Correlation Structure Between Credit Spreads and the Term Structure of Interest Rates: The Effects of Inflation and Earnings Uncertainty", Olin School of Business DP.

Diggle, P., and R., Milne, (1983): "Bivariate Cox Processes: Some Models for Bivariate Spatial Point Patterns", Journal of the Royal Statistical Society B, 45, 11-21.

Duffie, D., (1998): "First-to-Default Valuation", Stanford University DP.

Duffie, D., and N., Gârleanu, (2001): "Risk and Valuation of Collateralized Debt Obligations", Stanford University DP.

Duffie, D., and R., Kan (1996): "A Yield Factor Model of Interest Rates", Mathematical Finance, 6, 379-406.

Duffie, D., and D., Lando, (2001): "Term Structures of Credit Spreads with Incomplete Accounting Information", Econometrica, 69, 633-664.

Duffie, D., and K., Singleton, (1998): "Simulating Correlated Defaults", Stanford University DP.

Duffie, D., and K., Singleton, (1999): "Modelling Term Structures of Defaultable Bonds", Review of Financial Studies, 12, 687-720.

El Karoui, N., Frachot, A., and H., Geman, (1998): "On the Behaviour of Long Zero-Coupon Rates in a No Arbitrage Framework", Review of Derivatives Research, 1, 355-369.

Elliot, R., Jeanblanc, M., and M., Yor, (2000): "On Models of Default Risk", Mathematical Finance, 10, 179-195.

Florens, J.-P., and D., Fougere (1996): "Noncausality in Continuous Time", Econometrica, 64, 1195-1212.

Fons, J. (1994): "Using Default Rates to Model the Term Structure of Credit Risk", Financial Analysts Journal, September/October, 25-32.

Foulcher, S., Gouriéroux, C., and A., Tiomo, (2003): "La structure par terme de taux de défauts et ratings", Banque de France DP.

Frey, R., and A., McNeil (2001): "Modelling Dependent Defaults", Working Paper, ETH Zurich.

Gagliardini, P., and C., Gouriéroux (2002): "Constrained Nonparametric Dependence with Application to Finance", CREST and USI DP.

Genest, C., and R. J., Mc Kay (1986): "Copules Archimediennes et familles de lois bidimensionnelles dont les marges sont données", Can. J. Statistics, 14, 145-159.

Gouriéroux, C., and A., Monfort (1979): "On the Characterization of a Joint Probability Distribution by Conditional Distributions", Journal of Econometrics, 10, 115-118.

Gouriéroux, C., and A., Monfort (2002): "Equidependence in Qualitative and Duration Models", CREST DP.

Gouriéroux, C., and A., Monfort (2003): "Age and Term Structure in Duration Models", CREST DP.

Gouriéroux, C., Monfort, A., and V., Polimenis (2003): Affine Models for Credit Risk", CREST DP:

Griffiths, R., and R., Milne (1978): "A Class of Bivariate Poisson Processes", Journal of Multivariate Analysis, 8, 380-395.

Jarrow, R., Lando, D., and S., Turnbull, (1997): "A Markov Model for the Term Structure of Credit Risk Spread", Review of Financial Studies, 10, 481-523.

Jarrow, R., Lando, D., and F., Yu, (2001): "Default Risk and Diversification: Theory and Applications", Cornell University DP.

Jarrow, R., and S., Turnbull (2000): "The Intersection of Market and Credit Risk", Journal of Banking and Finance, 24, 271-299.

Jarrow, R., and F., Yu (2001): "Counterparty Risk and the Pricing of Defaultable Securities", Journal of Finance, 56, 1756-1799.

Joe, H., (1997): *Multivariate Models and Dependence Concepts*, Monograph on Statistics and Applied Probability, 73, Chapman & Hall.

Kalbfleisch, J., and R., Prentice (1980): *The Statistical Analysis of Failure Time Data*, Wiley, New-York.

Lando, D. (1994): "Three Essays on Contingent Claims Pricing", Ph.D. Dissertation, Cornell University.

Lando, D., (1997): "Modelling Bonds and Derivatives with Default Risk", in Dempster, M., and Pliska, S. eds., *Mathematics of Derivative Securities*, Cambridge University Press.

Lando, D., (1998): "On Cox Processes and Credit Risky Securities", Review of Derivatives Research, 2, 99-120.

Li, D. X., (2000): "On Default Correlation: a Copula Function Approach", Journal of Fixed Income, 9, 43-54.

Lucas, D., (1995): "Default Correlation and Credit Analysis", Journal of Fixed Income", 11, 76-87.

Merton, R., (1974): "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", Journal of Finance, 29, 449-470.

Oakes, D., (1982): "A Model for Association in Bivariate Survival Data", JRSS B, 44, 414-422.

Oakes, D. (1989): "Bivariate Survival Models Induced by Frailties", Journal of the American Statistical Association, 84, 487-493.

Rutkowski, M., (1999): "On Models of Default Risk", Warsaw University of Technology DP.

Schonbucher, P., (1998): ”Term Structure of Defaultable Bonds”, Review of Derivatives Research, 2, 161-192.

Schonbucher, P., (2003): "Information-Driven Default Contagion", ETHZ Working Paper.

Schonbucher, P., and D., Schubert (2001): ”Copula Dependent Default Risk in Intensity Models”, Bonn University DP.

Schwartz, T., (1998): ”Estimating the Term Structure of Corporate Debt”, Review of Derivatives Research, 2, 193-230.

Van den Berg, G. (1997): "Association Measures for Durations in Bivariate Hazard Models", Journal of Econometrics, 79, 225-245.

Van den Berg, G. (2001): "Duration Model, Specification, Identification and Multiple Durations", in Heckman, J., and E., Leamer, eds, *Handbook of Econometrics*, Vol. 5, Amsterdam, North-Holland.

Yu, F., (2003): "Correlated Defaults in Reduced-Form Models", University of California Irvine Working Paper.

Zhou, C., (2001): ”The Term Structure of Credit Spreads with Jump Risk”, Journal of Banking and Finance, 25, 2015-2040.

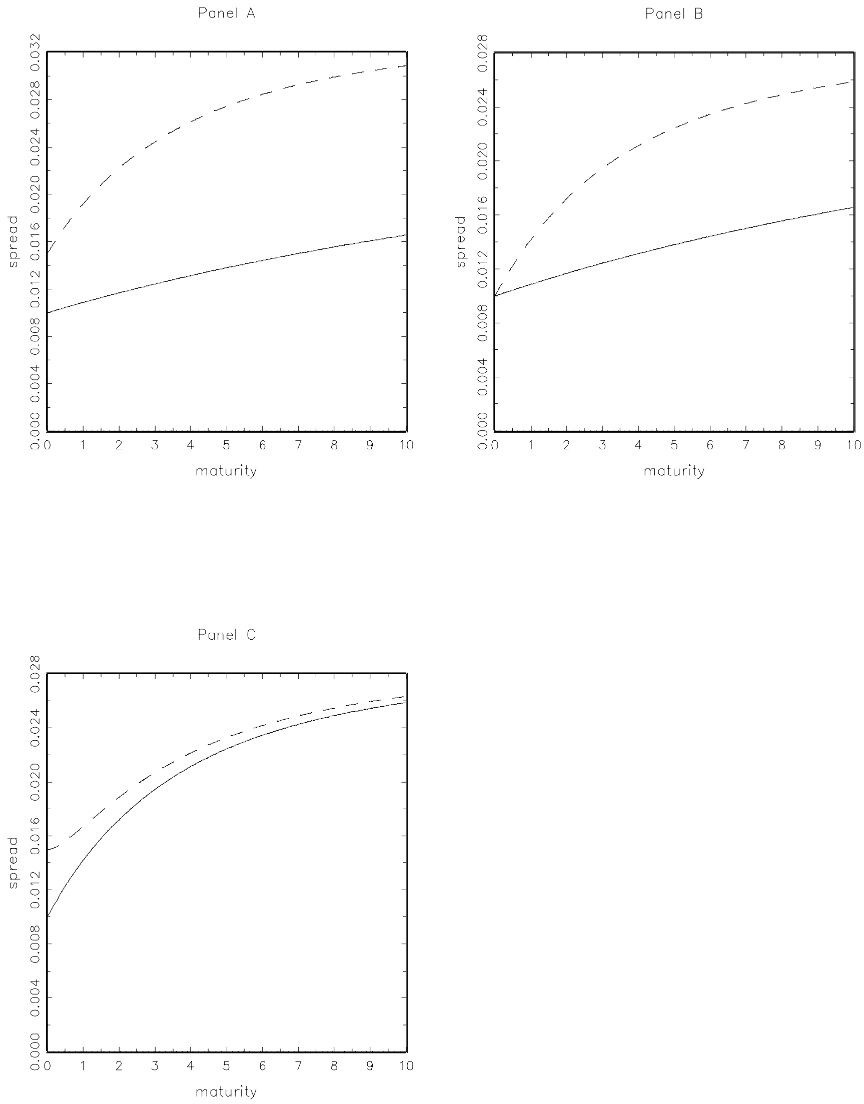

Figure 1: Term structure of interest rates associated with firm 1 when the second firm is still alive (solid line), and when the second firm defaulted earlier (dashed line). In Panel A the two curves differ at all term, in Panel B they differ in the long term, finally in panel C they differ in the short term but coincide in the long term.

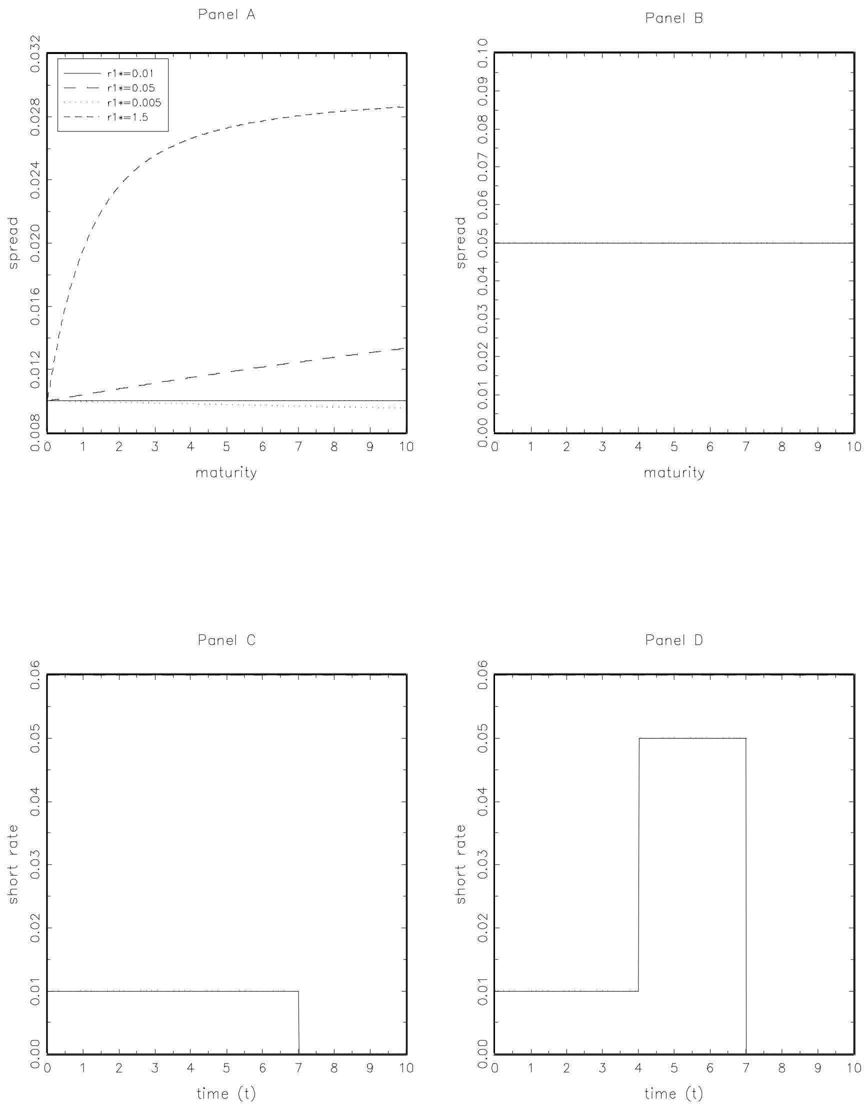

Figure 2: Constant intensities. In the upper panels we report the term structure associated with firm 1: when both firms are still alive, for the parameters $r_1 = 0.01$, $r_2 = 0.02$ and different values of r_1^* (Panel A), and when firm 2 has defaulted earlier, for the parameters $r_1 = 0.01$, $r_2 = 0.02$, $r_1^* = 0.05$ (Panel B). In the lower panels we report the short term spreads of firm 1 for the parameters $r_1 = 0.01$, $r_2 = 0.02$, $r_1^* = 0.05$: when firm 2 defaults after firm 1 $[Y_2 > Y_1 = 7]$ in Panel C, respectively before $[Y_2 = 4, Y_1 = 7]$ in Panel D.

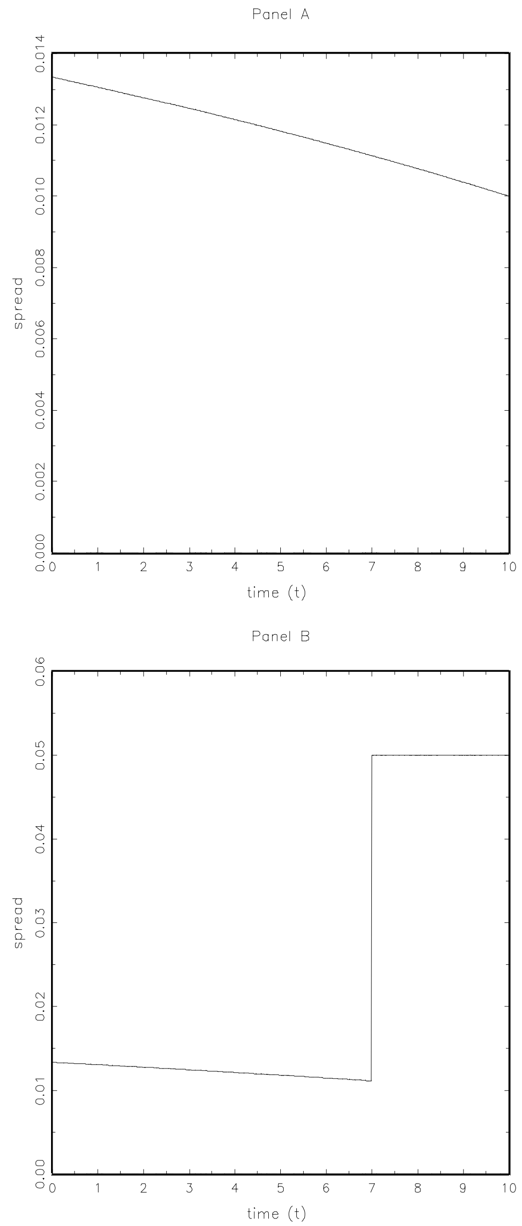

Figure 3: Constant intensities. Interest rate spread for a zero-coupon bond with maturity $t = H = 10$ issued by firm 1, for the parameters $r_1 = 0.01$, $r_2 = 0.02$, $r_1^* = 0.05$: when both firms default after H in Panel A, and when firm 2 defaults before H $[Y_2 = 7]$ in Panel B.

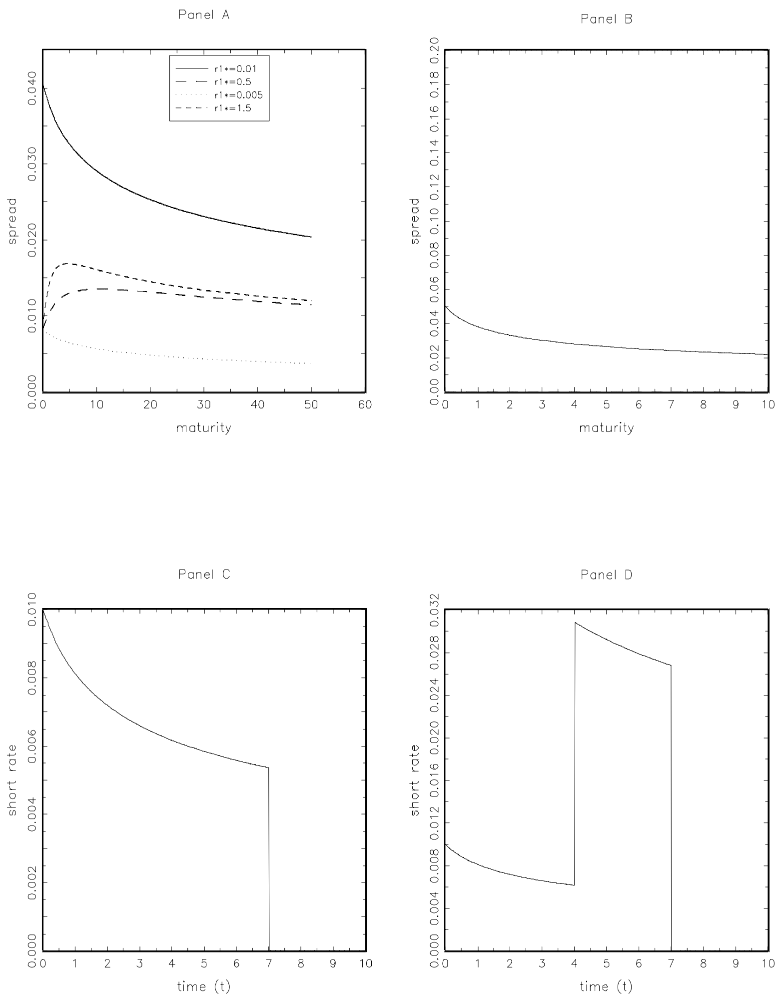

Figure 4: Model with proportional hazard for the parameters $r_1 = 0.01$, $r_2 = 0.02$, and a baseline hazard $\lambda_0(t) = 1/(1+t)^{0.3}$. In the upper panels we report the term structure associated with firm 1 at time $t = 1$: when both firms are still alive, for different values of r_1^*, in Panel A, and when firm 2 defaulted earlier, for the parameter $r_1^* = 0.05$, in Panel B. In the lower panels we report the short term spread associated with firm 1 for the parameter $r_1^* = 0.05$: when firm 2 defaults after firm 1 $[Y_2 > Y_1 = 7]$ in Panel C, respectively before $[Y_1 = 7, Y_2 = 4]$ in Panel D.

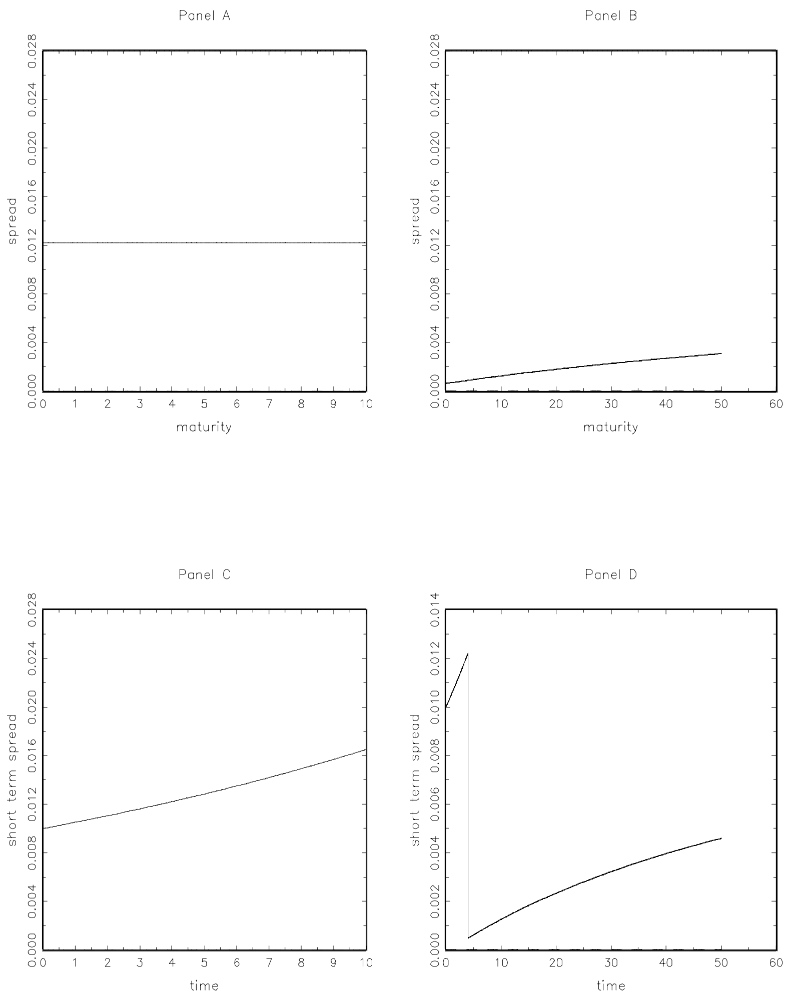

Figure 5: Flat term structures with intensities $\lambda_i(t) = r_i \exp(\beta_i t)$, $i = 1, 2$, $r_1 = 0.01$, $r_2 = 0.05$, $\beta_1 = 0.05$, $\beta_2 = 0.01$. In Panels A and B we report the term structure associated with firm 1 at time $t = 4$ when both firms are still alive, and at time $t = 5$ when firm 2 has defaulted at $t - k = 4$, respectively. In Panels C and D we report the short term spread of firm 1 when both firms are still alive, and when firm 2 defaults at $t - k = 4$, respectively.