Efficient Derivative Pricing by the Extended Method of Moments

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Introduction

The goal of the paper:

To estimate the pricing operator at a given date by using cross-sectional data on option prices and historical data on underlying asset returns

The pricing problem:

- The investor at date t_0 wants to estimate the current price

$$c_{t_0}(h, k)$$

  of a European derivative with time-to-maturity h and moneyness strike k that is not actively traded on the market

- The investor has data on
  - a cross-section of n current option prices $c_{t_0}(h_j, k_j), \; j = 1, ..., n$
  - a time series of T daily returns of the underlying asset

Contributions of the paper

- Introduce the Extended Method of Moments (XMM):

  An extension of the Generalized Method of Moments (GMM) to accommodate a more general set of moment restrictions for option pricing

  Account for the fact that the trading activity on each single derivative is much smaller than on the underlying index, and the characteristics of actively traded options vary over time

- Provide a semi-parametric estimator of the pricing operator at a given day

- The XMM estimators of risk premia and option prices are consistent for a large number T of historical observations on underlying asset returns and a finite number n of cross-sectionally observed option prices

- The XMM estimator outperforms the traditional cross-sectional calibration approach for S&P 500 option data

Activity on the S&P 500 index option market

The Chicago Board Options Exchange (CBOE) enhances the market of derivatives on the S&P 500 index by periodic issuing of new option contracts

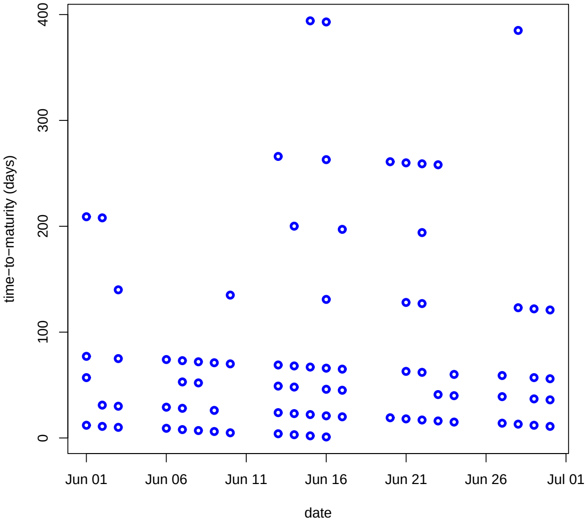
	Jan	*Feb*	*Mar*	*Apr*	*Mai*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*	...	*Mar*	...
Jan	$1m$	$2m$	$3m$			$6m$			$9m$			$12m$			
Feb		$1m$	$2m$	$3m$		$5m$			$8m$			$11m$			
Mar			$1m$	$2m$	$3m$	$4m$			$7m$			$10m$			
Apr				$1m$	$2m$	$3m$			$6m$			$9m$		$12m$	
Mai					$1m$	$2m$	$3m$		$5m$			$8m$		$11m$	
Jun						$1m$	$2m$	$3m$	$4m$			$7m$		$10m$	
Jul							$1m$	$2m$	$3m$			$6m$		$9m$	
Aug								$1m$	$2m$	$3m$		$5m$		$8m$	
Sep									$1m$	$2m$	$3m$	$4m$		$7m$	
Oct										$1m$	$2m$	$3m$		$6m$	
Nov											$1m$	$2m$		$5m$	
Dec												$1m$		$4m$	

New 12m options are issued when the old ones attain 9m-to-maturity

For any time-to-maturity, options are issued for a limited number of strikes

Highly traded S&P 500 options: times-to-maturity

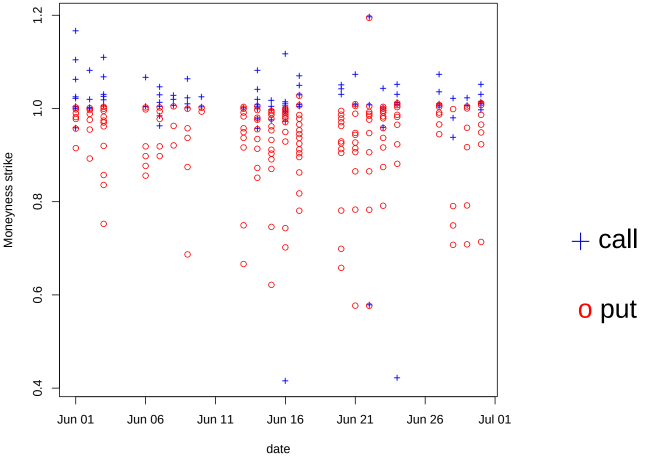
We consider S&P 500 options with daily trading volume larger than 4000 contracts in June 2005



Long times-to-maturity

are rarely actively traded!

Highly traded S&P 500 options: moneyness strikes



The number of highly traded options in a given trading day is rather small!

The number of options, the times-to-maturity and the moneyness strikes vary from one trading day to the other!

Calibration based on current derivative prices

- The issuing cycle and the trading activity imply that options' returns are rarely observable and potentially nonstationary!

- Standard methodology to circumvent these difficulties: daily calibration of the pricing operator using the cross-section of current option prices

  At date t_0:
  $$\hat{\theta}_{t_0} = \arg\min_{\theta} \sum_{j=1}^{n} [c_{t_0}(h_j, k_j) - c_{t_0}(h_j, k_j; \theta)]^2$$

  [Bakshi et al. (1997), Jackwerth (2000), Ait-Sahalia, Duarte (2003), Bondarenko (2003), Stutzer (1996), Jackwerth, Rubinstein (1996)]

- Drawbacks:
  - Estimated parameters $\hat{\theta}_{t_0}$ are erratic over time
  - Approximated prices $\hat{c}_{t_0}(h_j, k_j) = c_{t_0}(h_j, k_j; \hat{\theta}_{t_0})$ differ from observed prices for highly traded options
  - Estimates are not very accurate when n is small

Semi-parametric pricing

- X_t is the vector of observable state variables: a Markov process in $\mathcal{X} \subset \mathbb{R}^d$
- Semi-parametric specification:
  - the historical transition pdf $h(x_t|x_{t-1})$ of process X_t is left unconstrained
  - the stochastic discount factor $M_{t,t+1} = m\left(X_{t+1};\theta\right)$ is parameterized by $\theta \in \mathbb{R}^p$
- The price at date t of a European call option with time-to-maturity h and moneyness strike k is

$$c_t(h,k) = E\left[M_{t,t+h}(\theta)\left(\exp R_{t,h} - k\right)^+ |X_t\right]$$

- The goal is to estimate the pricing operator $(h,k) \rightarrow c_{t_0}(h,k)$ at a given date t_0
- Data consists in
  - Cross-section of n derivative prices $c_{t_0}(h_j,k_j), j = 1,...,n$ observed at t_0
  - Time-series of T observations of the state variables X_t before t_0

The moment restrictions

- The no-arbitrage constraints concerning the underlying asset:

$$E\left[M_{t,t+1}(\theta)\exp r_{t+1}|\, X_t = x\right] = 1 \qquad \forall x \in \mathcal{X}$$

$$\Leftrightarrow: \qquad E\left[g(Y;\theta)|X = x\right] = 0 \qquad \forall x \in \mathcal{X} \tag{1}$$

- The no-arbitrage constraints concerning the n observed derivatives at t_0:

$$c_{t_0}(h_j, k_j) = E\left[M_{t,t+h_j}(\theta)(\exp R_{t,h_j} - k_j)^+|X_t = x_{t_0}\right] \qquad j = 1,\ldots,n$$

$$\Leftrightarrow: \qquad E\left[\tilde{g}(Y;\theta)|X = x_0\right] = 0 \qquad x_0 \equiv x_{t_0} \tag{2}$$

- The conditional moment restrictions (1) are uniform since they hold for all values $x \in \mathcal{X}$

- The conditional moment restrictions (2) are local since they hold for the given value x_0 only $\rightarrow$ the key difference compared to standard GMM!

  Total set of $n+1$ local moment restrictions at t_0 given by $g_2 = (g', \tilde{g}')'$

Information-based GMM

- We need an estimator of both the sdf parameter θ and the historical transition pdf $f(y|x)$ to estimate the state price density!
- Related to the information-based GMM [Kitamura, Stutzer (1997), Imbens, Spady, Johnson (1998)]
- The kernel estimator of the historical transition pdf is

$$\hat{f}(y|x) = \frac{1}{h_T^d} \sum_{t=1}^{T} K\left(\frac{y_t - y}{h_T}\right) K\left(\frac{x_t - x}{h_T}\right) / \sum_{t=1}^{T} K\left(\frac{x_t - x}{h_T}\right)$$

  where K is the kernel and h_T is the bandwidth

- Select the conditional pdf that is the closest to $\hat{f}(y|x)$ and satisfies the moment restrictions (1) and (2)

The XMM estimator

The XMM estimator $\left(\hat{f}^*(\cdot|x_0), \hat{f}^*(\cdot|x_1), ..., \hat{f}^*(\cdot|x_T), \widehat{\theta}\right)$ *consists of the functions* $f_0, f_1, ..., f_T$ *and the parameter value* θ *that minimize*

$$L_T = \frac{1}{T}\sum_{t=1}^{T}\int \frac{\left[\widehat{f}(y|x_t) - f_t(y)\right]^2}{\widehat{f}(y|x_t)} dy + h_T^d \int \log\left[\frac{f_0(y)}{\widehat{f}(y|x_0)}\right] f_0(y) dy$$

subject to the constraints

$$\int f_t(y)dy = 1, \quad t = 1, ..., T \qquad \int f_0(y)dy = 1$$

$$\int g(y;\theta) f_t(y)dy = 0, \quad t = 1, ..., T \qquad \int g_2(y;\theta) f_0(y)dy = 0$$

- The chi-square criterion evaluated at the sample points allows for computation of $\hat{\theta}$ by parametric optimization
- The KLIC criterion evaluated at x_0 ensures a positive estimated state price density at t_0

The XMM estimator

XMM estimator of the historical conditional pdf given x_0

$$\widehat{f}^*(y|x_0) = \frac{\exp\left(\hat{\lambda}' g_2(y;\hat{\theta})\right)}{\widehat{E}\left[\exp\left(\hat{\lambda}' g_2(\hat{\theta})\right)|x_0\right]}\widehat{f}(y|x_0), \quad y \in \mathcal{Y}$$

where the Lagrange multiplier $\hat{\lambda} \in \mathbb{R}^{n+1}$ is s.t. $\widehat{E}\left[g_2(\hat{\theta})\exp\left(\hat{\lambda}' g_2(\hat{\theta})\right)|x_0\right] = 0$

XMM estimator of the derivative price $c_{t_0}(h,k)$

$$\hat{c}_{t_0}(h,k) = \int M_{t_0,t_0+h}(\hat{\theta})\left(\exp R_{t_0,h} - k\right)^+ \hat{f}^*(y|x_0)\, dy$$

for any time-to-maturity h and moneyness strike k

Estimator $\hat{c}_{t_0}(h,k)$ is equal to the observed price $c_{t_0}(h_j,k_j)$ for $h = h_j$ and $k = k_j$

Application to S&P 500 options

Compare XMM estimation and cross-sectional calibration on S&P 500 options with daily trading volume larger than 4000 contracts in June 2005

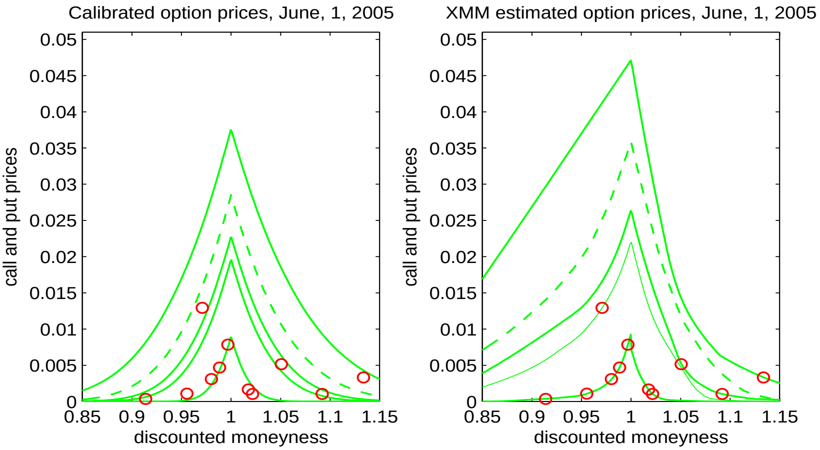
Cross-sectional calibration

- Parametric pricing formula from the risk-neutral distribution of a stochastic volatility model
- Stochastic volatility σ_t^2 follows a discrete-time Heston (1993) model

XMM estimation

- State variable vector: $X_t = (r_t, \sigma_t^2)'$ where σ_t^2 is the realized volatility of the S&P 500 index
- Parametric sdf: $M_{t,t+1}(\theta) = \exp\left(-\theta_1 - \theta_2\sigma_{t+1}^2 - \theta_3\sigma_t^2 - \theta_4 r_{t+1}\right)$
- XMM estimator is computed for each trading day t_0 using current option data and previous $T = 1000$ daily observations of the state variables

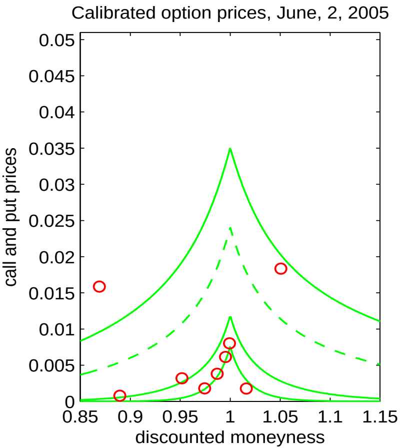
Estimated call/put price functions: June 1, 2005

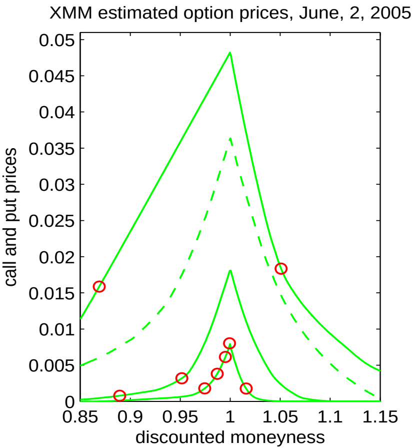


Highly traded times-to-maturity (solid): 12, 57, 77, 209-day. Non traded (dashed): 120-day

- XMM estimated prices coincide with market prices for highly traded options!
- Discrepancies for calibrated prices can be large

Estimated call/put price functions: June 2, 2005

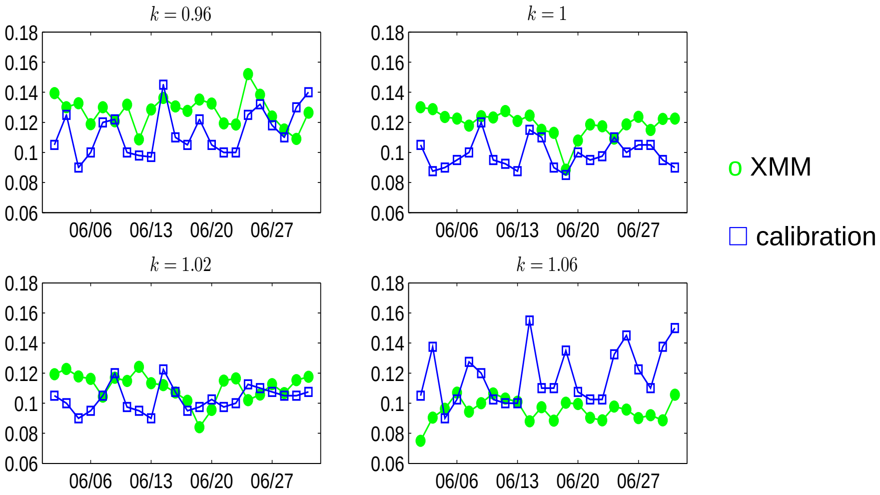




Highly traded times-to-maturity (solid): 11, 31, 208-day. Non traded (dashed): 119-day

- XMM yields skewed option price curves: it captures the leverage effect and its term structure!

Time-series of estimated implied volatilities



- XMM estimated implied volatilities are more stable in time!

Large sample properties of the XMM estimator

- Estimators $\hat{\theta}$ and $\hat{c}_{t_0}(h,k)$ are consistent and asymptotically normal as $T \to \infty$ and the number of options n is fixed

- Estimated option prices $\hat{c}_{t_0}(h,k)$ converge at rate $\sqrt{Th_T^d}$

  $\Leftrightarrow$ convergence rate for nonparametric estimation of conditional expectation given $X = x_0$

- Linear transformations of θ that are

  $\rightarrow$ identifiable from uniform moment restrictions (1) from underlying asset $\rightarrow$ parametric convergence rate $\sqrt{T}$

  $\rightarrow$ unidentifiable from uniform moment restrictions (1) from underlying asset $\rightarrow$ nonparametric convergence rate $\sqrt{Th_T^d}$

- Example of stochastic volatility model: risk premium parameter for stochastic volatility converges at rate $\sqrt{Th_T^d}$

Link with weak instruments

- The nonstandard large sample properties of XMM are related to a weak instrument problem

- The local conditional moment restriction

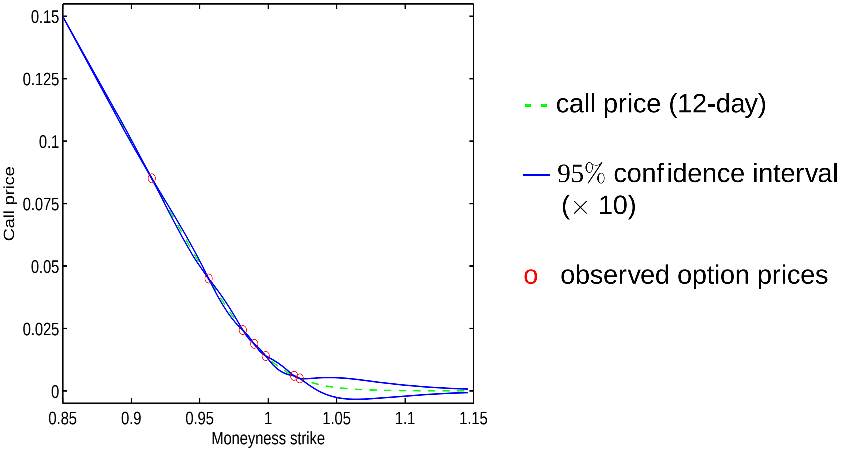
$$
\begin{aligned}
E\left[\tilde{g}(Y;\theta)|X=x_0\right] &\simeq E\left[\tilde{g}(Y;\theta)\frac{1}{h_T^d}K\left(\frac{X-x_0}{h_T}\right)\right]\frac{1}{f_X(x_0)} \\
&= E\left[Z\tilde{g}(Y;\theta)\right]
\end{aligned}
$$

  is equivalent to an unconditional moment restriction with the "weak" instrument

$$
Z=\frac{1}{h_T^d}K\left(\frac{X-x_0}{h_T}\right)\frac{1}{f_X(x_0)}
$$

Kernel nonparametric efficiency bound

- XMM estimator of option prices attains the efficiency bound in a class of kernel-GMM estimators with optimal instruments and weighting matrix



- The width of the CI depends on moneyness strike k and is equal to zero when k is the moneyness strike of an observed option

Concluding remarks

- The traditional literature on joint estimation of historical and risk-neutral parameters adopts ML and GMM and
  - either assumes that risk premia are identifiable from historical dynamics alone [e.g. Hansen, Jagannathan (1997), Stock, Wright (2000)]
  - or relies on a few artificial time-series of option prices [Duan (1994), Chernov, Ghysels (2000), Pan (2002), Eraker (2004)]
- XMM extends GMM to accommodate the pricing restrictions from both historical data and cross-sectionally observed option prices
- The XMM estimator is consistent for a finite number of observed derivative prices, even when some sdf parameters are not full-information identifiable
- The new XMM-based calibration method outperforms the traditional calibration approach in the application to S&P 500 options