G Supplementary materials

In this section we give the proofs of the technical lemmas used in the paper *Semi-Parametric Estimation of American Option Prices*, by P. Gagliardini and D. Ronchetti.

G.1 Proof of Lemma 1

By the triangular inequality we get

$$\left\|\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} \le \left\|\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} + \left\|\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty}. \tag{G.1}$$

The first term in the RHS of Inequality (G.1) is the supremum norm on set $\mathcal{Y}_T$ of the function $\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]$, which can be written as

$$\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta](\tilde{y})-\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta](\tilde{y}) = \int_{\mathcal{X}_T} m(x;\theta)e^r\left[\hat{\varphi}_\theta-\varphi_\theta\right](\tilde{k}e^{-r},x)\hat{f}(x|\tilde{x})dx,$$

for any $\tilde{y} := [\tilde{k}\ \tilde{x}']'$ in $\mathcal{Y}$. Then we have

$$\begin{aligned}\left\|\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} &\le \|\hat{\varphi}_\theta-\varphi_\theta\|_{\mathcal{Y}'_T,\infty}\sup_{\tilde{x}\in\mathcal{X}_T}\int_{\mathcal{X}_T}\left|m(x;\theta)e^r\left[\frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}+1\right]f_0(x|\tilde{x})\right|dx\\ &\le \|\hat{\varphi}_\theta-\varphi_\theta\|_{\mathcal{Y}'_T,\infty}\left[\sup_{x,\tilde{x}\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}\right|+1\right]e^b\sup_{x\in\mathcal{X}}\mathrm{E}\left[|m(X_{t+1};\theta)|\,|\,X_t=x\right],\end{aligned}$$

where $b>0$ is defined at point **iv)** of Section F.1 and $\Delta\hat{f} := \hat{f}-f_0$.

Lemma 3. *Under Assumptions A 1, A 2 and A 4-6,* $\sup_{x,\tilde{x}\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}\right| = o_p(1)$.

Proof. See Section G.3. □

From the Cauchy-Schwarz inequality, Assumption A 9, Lemma 3, and the condition $\sup_{\theta\in\Theta}\|\hat{\varphi}_\theta-\varphi_\theta\|_{\mathcal{Y}'_T,\infty} = o_p(1)$, we get

$$\sup_{\theta\in\Theta}\left\|\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} = O_p\left(\sup_{\theta\in\Theta}\|\hat{\varphi}_\theta-\varphi_\theta\|_{\mathcal{Y}'_T,\infty}\right) = o_p(1). \tag{G.2}$$

The second term in the RHS of Inequality (G.1) is the supremum norm on set $\mathcal{Y}_T$ of the function $\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]$ given by

$$\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta](\tilde{y})-\mathcal{E}_{\theta,f_0}[\varphi_\theta](\tilde{y}) = \int_{\mathcal{X}_T} m(x;\theta)e^r\varphi_\theta(\tilde{k}e^{-r},x)\Delta\hat{f}(x|\tilde{x})dx - \int_{\mathcal{X}_T^C} m(x;\theta)e^r\varphi_\theta(\tilde{k}e^{-r},x)f_0(x|\tilde{x})dx.$$

Then by the triangle inequality we have

$$\left\|\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} \leq \sup_{x,\tilde{x}\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}\right|\sup_{\tilde{y}\in\mathcal{Y}}\int_{\mathcal{X}_T} m(x;\theta)e^r\left|\varphi_\theta(\tilde{k}e^{-r},x)\right| f_0(x|\tilde{x})dx + \sup_{\tilde{y}\in\mathcal{Y}}\int_{\mathcal{X}_T^C} m(x;\theta)e^r\left|\varphi_\theta(\tilde{k}e^{-r},x)\right| f_0(x|\tilde{x})dx. \tag{G.3}$$

By the Cauchy-Schwarz inequality we get

$$\int_{\mathcal{X}_T} m(x;\theta)e^r\left|\varphi_\theta(\tilde{k}e^{-r},x)\right| f_0(x|\tilde{x})dx \leq e^b\left(\mathrm{E}\left[m(X_{t+1};\theta)^2\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{2}}\left(\mathrm{E}\left[|\varphi_\theta(\tilde{k}e^{-r_{t+1}},X_{t+1})|^2\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{2}}.$$

Similarly, we get the counterpart of this inequality for the set-theoretical complement $\mathcal{X}_T^C$ of the domain of integration:

$$\int_{\mathcal{X}_T^C} m(x;\theta)e^r\left|\varphi_\theta(\tilde{k}e^{-r},x)\right| f_0(x|\tilde{x})dx \leq e^b\left(\mathrm{E}\left[m(X_{t+1};\theta)^2\mathbf{1}_{\mathcal{X}_T^C}(X_{t+1})\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{2}}\left(\mathrm{E}\left[|\varphi_\theta(\tilde{k}e^{-r_{t+1}},X_{t+1})|^2\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{2}}.$$

Let us focus on the square of the second term in the RHS of the previous inequality. By the Hölder inequality we get

$$\mathrm{E}\left[m(X_{t+1};\theta)^2\mathbf{1}_{\mathcal{X}_T^C}(X_{t+1})\middle| X_t=\tilde{x}\right] \leq \left(\mathrm{E}\left[|m(X_{t+1};\theta)|^{2\bar{p}}\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{\bar{p}}}\underbrace{\left(\mathrm{E}\left[\mathbf{1}_{\mathcal{X}_T^C}(X_{t+1})\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{\bar{q}}}}_{=\left(\mathbb{P}\left[X_{t+1}\in\mathcal{X}_T^C\middle| X_t=\tilde{x}\right]\right)^{\frac{1}{\bar{q}}}},$$

where $\bar{p},\bar{q}>1$ are such that $\frac{1}{\bar{p}}+\frac{1}{\bar{q}}=1$. Thus, from Inequality (G.3) we get

$$\left\|\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty} \leq e^b\left\{\sup_{x,\tilde{x}\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}\right|\sup_{x\in\mathcal{X}}\left(\mathrm{E}\left[m(X_{t+1};\theta)^2\middle| X_t=x\right]\right)^{\frac{1}{2}} + \sup_{x\in\mathcal{X}_T}\left(\mathrm{E}\left[|m(X_{t+1};\theta)|^{2\bar{p}}\middle| X_t=x\right]\right)^{\frac{1}{\bar{p}}}\sup_{x\in\mathcal{X}_T}\left(\mathbb{P}\left[X_{t+1}\in\mathcal{X}_T^C\middle| X_t=x\right]\right)^{\frac{1}{\bar{q}}}\right\} \cdot \sup_{y\in[e^{-a},e^a]\times\mathcal{X}}\left(\mathrm{E}\left[|\varphi_\theta(Y_{t+1})|^2\middle| Y_t=y\right]\right)^{\frac{1}{2}}.$$

Let us choose $\bar{p}$ such that $2\bar{p}=2+\delta$, where $\delta>0$ is defined in Assumption A 9. From Lemma 3, Assumptions A 4 and A 9 and Inequality (F.2) we get

$$\sup_{\theta\in\Theta}\left\|\mathcal{E}_{\theta,\hat{f}}[\varphi_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty}=o_p(1). \tag{G.4}$$

Thus, from Inequality (G.1) and Equations (G.2) and (G.4), we get $\sup_{\theta\in\Theta}\left\|\mathcal{E}_{\theta,\hat{f}}[\hat{\varphi}_\theta]-\mathcal{E}_{\theta,f_0}[\varphi_\theta]\right\|_{\mathcal{Y}_T,\infty}=o_p(1)$.

G.2 Proof of Lemma 2

We use the notation $\mathcal{E}=\mathcal{E}_{\theta,f_0}$ and $\mathcal{A}=\mathcal{A}_{\theta,f_0}$. Let $h\in\mathbb{N}$ and assume that

$$\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a-b},e^{a+b}]\times\mathcal{X}}}\mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+1})^2\middle|\,Y_t=y\right]<\infty. \tag{G.5}$$

Let us now prove Inequality (F.5). We write the $(h+1)$-fold application of operator $\mathcal{A}$ as in Equation (F.3):

$$\mathcal{A}^{h+1}[v(0,\cdot)]=v(0,\cdot)+\Big(\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)]-v(0,\cdot)\Big)^+.$$

Since $(t-s)^+\leq|t|+|s|$, for all $t,s\in\mathbb{R}$, we get

$$\mathcal{A}^{h+1}[v(0,\cdot)]^2\leq\Big(2v(0,\cdot)+\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)]\Big)^2. \tag{G.6}$$

From Inequality (G.6) we get

$$\begin{aligned}\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\mathcal{A}^{h+1}[v(0,\cdot)](Y_{t+1})^2\middle|\,Y_t=y\right]&\leq 4A+\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\Big(\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)](Y_{t+1})\Big)^2\middle|\,Y_t=y\right]\\&\quad+4\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[v(0,Y_{t+1})\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)](Y_{t+1})\middle|\,Y_t=y\right],\end{aligned} \tag{G.7}$$

where $A:=e^{2(a+b)}$. We apply the Cauchy-Schwarz inequality and the Law of Iterated Expectations to the second term in the RHS of Inequality (G.7) to get

$$\begin{aligned}&\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\Big(\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)](Y_{t+1})\Big)^2\middle|\,Y_t=y\right]\\&=\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\left(\int_{\mathcal{X}}m(\tilde{x};\theta)e^{\tilde{r}}\mathcal{A}^h[v(0,\cdot)](k_{t+1}e^{-\tilde{r}},\tilde{x})f(\tilde{x}|X_{t+1})d\tilde{x}\right)^2\middle|\,Y_t=y\right]\\&\leq e^{2b}\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\mathrm{E}\left[m(X_{t+2};\theta)^2\middle|\,X_{t+1}\right]\mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2\middle|\,Y_{t+1}\right]\middle|\,Y_t=y\right]\\&\leq e^{2b}\,C_2\sup_{\substack{\theta\in\Theta\\ y\in[e^{-a},e^{a}]\times\mathcal{X}}}\mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2\middle|\,Y_t=y\right],\end{aligned} \tag{G.8}$$

where $C_2 := \sup_{\substack{\theta\in\Theta \\ x\in\mathcal{X}}} \mathrm{E}\left[|m(X_{t+1};\theta)|^2 \middle| X_t = x\right]$ is finite from Assumption A 9 **(ii)**. We apply the Cauchy-Schwarz inequality to the last term in the RHS of Inequality (G.7) and we make use of Inequalities (G.8):

$$\begin{aligned}
&\sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[v(0,Y_{t+1})\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)](Y_{t+1}) \middle| Y_t = y\right] \\
&\leq \left(A \sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\left(\mathcal{E}\circ\mathcal{A}^h[v(0,\cdot)](Y_{t+1})\right)^2 \middle| Y_t = y\right]\right)^{\frac{1}{2}} \\
&\leq \left(e^{2b}\, C_2 A \sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2 \middle| Y_t = y\right]\right)^{\frac{1}{2}}. \qquad \text{(G.9)}
\end{aligned}$$

By grouping Inequalities (G.5) and (G.7)-(G.9) we get

$$\begin{aligned}
\sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^{h+1}[v(0,\cdot)](Y_{t+1})^2 \middle| Y_t = y\right] &\leq 4A + e^{2b}\, C_2 \sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^{h}[v(0,\cdot)](Y_{t+2})^2 \middle| Y_t = y\right] \\
&\quad + \left(e^{2b}\, C_2 A \sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2 \middle| Y_t = y\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

By using the Law of Iterated Expectations and $k_{t+1} = k_t e^{-r_{t+1}}$ with $|r_{t+1}| \leq b$, we have:

$$\sup_{\substack{\theta\in\Theta \\ y\in[e^{-a},e^{a}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2 \middle| Y_t = y\right] \leq \sup_{\substack{\theta\in\Theta \\ y\in[e^{-a-b},e^{a+b}]\times\mathcal{X}}} \mathrm{E}\left[\mathcal{A}^h[v(0,\cdot)](Y_{t+2})^2 \middle| Y_{t+1} = y\right] < \infty,$$

from Inequality (G.5). The conclusion follows.

G.3 Proof of Lemma 3

Let us consider the kernel estimator $\hat{f}_X$ of the stationary pdf f_X of X_t defined in Equation (4.2) and the kernel estimator $\hat{f}_Z$ of the stationary pdf f_Z of $[X_t'\ X_{t-1}']'$ defined by

$$\hat{f}_Z(x,\tilde{x}) = \frac{1}{Th_T^{2d}} \sum_{t=2}^{T} K\left(\frac{X_t - x}{h_T}\right) K\left(\frac{X_{t-1} - \tilde{x}}{h_T}\right).$$

Let us define $\Delta\hat{f}_Z(x,\tilde{x}) := \hat{f}_Z(x,\tilde{x}) - f_Z(x,\tilde{x})$ and $\Delta\hat{f}_X(x) := \hat{f}_X(x) - f_X(x)$. From the uniform convergence of the kernel density estimation (see, e.g., Hansen [2008]) and Assumptions A 1, A 2, A 5 and A 6 we have

$$\sup_{x,\tilde{x}\in\mathcal{X}_T} \left|\Delta\hat{f}_Z(x,\tilde{x})\right| = O_p\left(\sqrt{\frac{\log(T)}{Th_T^{2d}}} + h_T^{\rho}\right), \quad \sup_{x\in\mathcal{X}_T} \left|\Delta\hat{f}_X(x)\right| = O_p\left(\sqrt{\frac{\log(T)}{Th_T^{d}}} + h_T^{\rho}\right). \qquad \text{(G.10)}$$

From Assumptions A 4 and A 6 and Equations (G.10), we have

$$\begin{aligned} \sup_{x,\tilde{x}\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}_Z(x,\tilde{x})}{f_Z(x,\tilde{x})}\right| &= O_p\left((\log(T))^{c_1}\left(\sqrt{\frac{\log(T)}{Th_T^{2d}}}+h_T^{\rho}\right)\right)=o_p(1),\\ \sup_{x\in\mathcal{X}_T}\left|\frac{\Delta\hat{f}_X(x)}{f_X(x)}\right| &= O_p\left((\log(T))^{c_2}\left(\sqrt{\frac{\log(T)}{Th_T^{d}}}+h_T^{\rho}\right)\right)=o_p(1). \end{aligned} \tag{G.11}$$

Since $f_0(x|\tilde{x})=f_Z(x,\tilde{x})/f_X(\tilde{x})$ and $\hat{f}(x|\tilde{x})=\hat{f}_Z(x,\tilde{x})/\hat{f}_X(\tilde{x})$ we get

$$\begin{aligned} \frac{\Delta\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})} &= \frac{\hat{f}(x|\tilde{x})}{f_0(x|\tilde{x})}-1=\frac{\hat{f}_Z(x,\tilde{x})}{\hat{f}_X(\tilde{x})f_0(x|\tilde{x})}-1=\frac{f_Z(x,\tilde{x})+\Delta\hat{f}_Z(x,\tilde{x})}{\left[f_X(\tilde{x})+\Delta\hat{f}_X(\tilde{x})\right]\dfrac{f_Z(x,\tilde{x})}{f_X(\tilde{x})}}-1\\ &= \frac{1+\dfrac{\Delta\hat{f}_Z(x,\tilde{x})}{f_Z(x,\tilde{x})}}{1+\dfrac{\Delta\hat{f}_X(\tilde{x})}{f_X(\tilde{x})}}-1=\frac{\dfrac{\Delta\hat{f}_Z(x,\tilde{x})}{f_Z(x,\tilde{x})}-\dfrac{\Delta\hat{f}_X(\tilde{x})}{f_X(\tilde{x})}}{1+\dfrac{\Delta\hat{f}_X(\tilde{x})}{f_X(\tilde{x})}}, \end{aligned} \tag{G.12}$$

for any $x,\tilde{x}\in\mathcal{X}_T$. From Equations (G.11) and (G.12) the conclusion follows.