# Supplementary material for

# Survival of Hedge Funds: Frailty vs Contagion

(November 2012)

This supplementary material provides in Section 1 the computation of the term structures of conditional first- and second-order moments of liquidation counts. In Section 2 we present the figures of the term structures of conditional volatility and overdispersion of liquidation counts under the three stressing scenarios described in Section 3.6 of the paper.

## 1 Term structures of conditional moments of liquidation counts

Let us consider the dynamic Poisson model with Autoregressive Gamma frailty (see Sections 2.3 and 3.4):

$$Y_{k,t} \sim \mathcal{P}(a_k + b_k F_t + c_k' Y_{t-1}), \quad k = 1, ..., K.$$

The size adjustments are $\gamma_{k,t} = 1$, for all $t$ and $k$. Since the joint process $(Y_t, F_t)$ is affine, the conditional mean and variance of vector $(Y_{t+1}', F_{t+1})'$ given $(\underline{Y_t}, \underline{F_t})$ are linear affine functions of vector $(Y_t', F_t)'$. This implies that the vector including the elements of $Y_t$, $F_t$, their squares and cross-terms:

$$\xi_t = (F_t, Y_t', F_t^2, F_t Y_t', vec(Y_t Y_t')')', \tag{b.1}$$

is such that the conditional expectation $E_t[\xi_{t+1}]$ is a linear affine function of $\xi_t$. [1] Therefore:

$$E_t[\xi_{t+1}] = E[\xi_t] + \Psi(\xi_t - E[\xi_t]), \tag{b.2}$$

for a matrix $\Psi$, given in Proposition A.1 below. The unconditional mean $E[\xi_t]$ can be computed from Proposition 1 and equation (a.3) in the paper:

$$E[\xi_t] = \left(1, \mu', 1 + \delta^{-1}, \Gamma', vec(M_2)'\right)', \tag{b.3}$$

where $\mu = (Id - C)^{-1}(a + b)$, $\Gamma = \delta^{-1}(Id - \rho C)^{-1} b + \mu$, and the matrix $M_2 = E[Y_t Y_t']$ is such that:

$$\begin{aligned} vec(M_2) &= (Id - C \otimes C)^{-1} vec\left[diag(\mu) + \delta^{-1} bb' + \delta^{-1}\rho C (Id - \rho C)^{-1} bb' \right. \\ &\quad \left. + \delta^{-1}\rho bb'(Id - \rho C')^{-1} C'\right] + \mu \otimes \mu, \end{aligned}$$

[1] We could use $vech(Y_t Y_t')$ instead of $vec(Y_t Y_t')$ to include in vector $\xi_t$ only the different elements of the symmetric matrix $Y_t Y_t'$. However, our choice simplifies the expression of matrix $\Psi$ in Proposition A.1.

where $\delta$ is the degree of freedom of the gamma stationary distribution of the frailty, $\rho$ is its first-order autocorrelation, and $\otimes$ denotes the Kronecker product. By the Law of Iterated Expectation, we deduce:

$$E_t[\xi_{t+\tau}] = E[\xi_t] + \Psi^\tau(\xi_t - E[\xi_t]), \tag{b.4}$$

for any horizon $\tau \geq 1$. This equation allows to compute the term structure of conditional first- and second-order moments of process $(Y_t, F_t)$. We use equation (b.4) to compute the term structures of expected liquidation counts in Figures 12-14 of the paper, and the term structures of volatility and overdispersion of liquidation counts in Figures 15-20 in Section 2 of this supplementary material.

**Proposition A.1:** *The matrix* $\Psi$ *is given by:*

$$\Psi = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ \rho b & C & 0 & 0 & 0 \\ \psi_{31} & 0 & \rho^2 & 0 & 0 \\ \rho a + \psi_{31} b & (1-\rho)C & \rho^2 b & \rho C & 0 \\ \psi_{51} & \psi_{52} & \rho^2 b \otimes b & \rho(b \otimes C + C \otimes b) & C \otimes C \end{pmatrix}, \tag{b.5}$$

*in a block form corresponding to the block form of vector* $\xi_t$ *in equation (b.1), where:*

$$\begin{aligned} \psi_{31} &= 2\rho(1-\rho)(1/\delta + 1), \\ \psi_{51} &= \rho[vec(diag(b)) + a \otimes b + b \otimes a] + \psi_{31} b \otimes b, \\ \psi_{52} &= \tilde{C} + [a + (1-\rho)b] \otimes C + C \otimes [a + (1-\rho)b], \end{aligned}$$

*and* $\tilde{C}$ *is the* $K^2 \times K$ *matrix whose* $i$*-th column is equal to* $vec(diag(c_i))$*, where* $c_i$ *is the* $i$*-th column of* $C$*.*

**Proof:** Let us compute the conditional expectation of the elements of vector $\xi_{t+1}$.

a) From Appendix 1 i), the conditional mean of $F_{t+1}$ given $(\underline{Y_t}, \underline{F_t})$ is:

$$E_t[F_{t+1}] = 1 + \rho(F_t - 1). \tag{b.6}$$

b) From Appendix 2 ii), the conditional mean of $Y_{t+1}$ given $(\underline{Y_t}, \underline{F_t})$ is:

$$E_t[Y_{t+1}] = a + bE_t[F_{t+1}] + CY_t = \mu + \rho b(F_t - 1) + C(Y_t - \mu). \tag{b.7}$$

c) Let us now consider the conditional mean of $F_{t+1}^2$ given $(\underline{Y_t}, \underline{F_t})$. From Appendix 1 i) we have:

$$V_t[F_{t+1}] = \nu^2\delta + 2\eta\nu^2 F_t = \frac{1-\rho^2}{\delta} + 2\frac{\rho(1-\rho)}{\delta}(F_t - 1), \tag{b.8}$$

where we use $\rho = \nu\eta$ and $\nu = (1-\rho)/\delta$. From equation (b.6) we get:

$$E_t[F_{t+1}^2] = 1/\delta + 1 + \rho^2(F_t^2 - 1/\delta - 1) + 2\rho(1-\rho)(1/\delta+1)(F_t - 1). \quad \text{(b.9)}$$

d) Let us now consider the conditional mean of $Y_{t+1}F_{t+1}$ given $(\underline{Y_t}, \underline{F_t})$. From Appendix 2 iii) we have:

$$E_t[Y_{t+1}F_{t+1}] = aE_t[F_{t+1}] + bE_t[F_{t+1}^2] + C(1-\rho+\rho F_t)Y_t. \quad \text{(b.10)}$$

From equations (b.6) and (b.9) we get:

$$\begin{aligned} E_t[Y_{t+1}F_{t+1}] &= a + (1/\delta+1)b + [\rho a + 2\rho(1-\rho)(1/\delta+1)b](F_t - 1) + \rho^2 b(F_t^2 - 1/\delta - 1) \\ &\quad +(1-\rho)CY_t + \rho CY_tF_t. \end{aligned}$$

We can write the conditional expectation $E_t[Y_{t+1}F_{t+1}]$ as a linear affine function of the zero-mean processes $F_t - 1$, $F_t^2 - 1/\delta - 1$, $Y_t - \mu$ and $Y_tF_t - \Gamma$. Then, the constant in such an equation must be the unconditional expectation $\Gamma$, and we get:

$$\begin{aligned} E_t[Y_{t+1}F_{t+1}] &= \Gamma + [\rho a + 2\rho(1-\rho)(1/\delta+1)b](F_t - 1) + \rho^2 b(F_t^2 - 1/\delta - 1) \\ &\quad +(1-\rho)C(Y_t - \mu) + \rho C(Y_tF_t - \Gamma). \end{aligned} \quad \text{(b.11)}$$

e) Finally, let us consider the conditional mean of $Y_{t+1}Y_{t+1}'$ given $(\underline{Y_t}, \underline{F_t})$. From Appendix 2 iii) we have:

$$\begin{aligned} E_t[Y_{t+1}Y_{t+1}'] &= diag(a + bE_t[F_{t+1}] + CY_t) + bb'V_t[F_{t+1}] \\ &\quad +(a + bE_t[F_{t+1}] + CY_t)(a + bE_t[F_{t+1}] + CY_t)'. \end{aligned}$$

From equations (b.6) and (b.8) we get:

$$\begin{aligned} E_t[Y_{t+1}Y_{t+1}'] &= diag(a + (1-\rho)b + \rho bF_t + CY_t) + \frac{1-\rho^2}{\delta}bb' + 2\frac{\rho(1-\rho)}{\delta}bb'(F_t - 1) \\ &\quad +[a + (1-\rho)b][a + (1-\rho)b]' + \rho[a + (1-\rho)b]b'F_t + [a + (1-\rho)b]Y_t'C' \\ &\quad +\rho b[a + (1-\rho)b]'F_t + \rho^2 bb'F_t^2 + \rho bF_tY_t'C' \\ &\quad +CY_t[a + (1-\rho)b]' + \rho CY_tF_tb' + CY_tY_t'C'. \end{aligned}$$

Therefore, by gathering in the RHS the terms with the zero-mean processes $F_t - 1$, $F_t^2 - 1/\delta - 1$, $Y_t - \mu$, $Y_tF_t - \Gamma$ and $Y_tY_t' - M_2$, we get:

$$\begin{aligned} E_t[Y_{t+1}Y_{t+1}'] &= M_2 + \{\rho[diag(b) + ab' + ba'] + 2\rho(1-\rho)(1/\delta+1)bb'\}(F_t - 1) \\ &\quad +diag[C(Y_t - \mu)] + C(Y_t - \mu)[a + (1-\rho)b]' + [a + (1-\rho)b](Y_t - \mu)'C' \\ &\quad +\rho^2 bb'(F_t^2 - 1/\delta - 1) + \rho C(Y_tF_t - \Gamma)b' + \rho b(Y_tF_t - \Gamma)'C' + C(Y_tY_t' - M_2)C'. \end{aligned}$$

Let us now compute the $vec$ of both sides of the equation, by using that $vec(ABC) = (C' \otimes A)vec(B)$ for conformable matrices $A, B, C$, $vec(ab') = b \otimes a$ for vectors $a, b$, and $vec(diag[C(Y_t - \mu)]) = \tilde{C}(Y_t - \mu)$ [see e.g. Magnus, Neudecker (2007)]. We get:

$$\begin{aligned} E_t[vec(Y_{t+1}Y'_{t+1} - M_2)] &= \{\rho[vec(diag(b)) + a \otimes b + b \otimes a] + 2\rho(1-\rho)(1/\delta + 1)b \otimes b\} (F_t - 1) \\ &+ \left(\tilde{C} + [a + (1-\rho)b] \otimes C + C \otimes [a + (1-\rho)b]\right) (Y_t - \mu) \\ &+ \rho^2 b \otimes b(F_t^2 - 1/\delta - 1) + \rho(b \otimes C + C \otimes b)(Y_t F_t - \Gamma) \\ &+ (C \otimes C)vec(Y_t Y'_t - M_2). \end{aligned} \tag{b.12}$$

From equations (b.6), (b.7), (b.9), (b.11) and (b.12) the conclusion follows. ■

# 2 Term structures of conditional volatilities and overdispersion of liquidation counts under stress scenarios

In Figures 15 and 16 we display the term structures of the volatility of the liquidation counts, and of their overdispersion, respectively, under stressing of the factor value (stress scenario S.1 defined in Section 3.6).

[Insert Figure 15: Term structure of liquidation volatility when stressing the current factor value]

[Insert Figure 16: Term structure of liquidation overdispersion when stressing the current factor value]

The term structures of volatility and overdispersion of the liquidation counts under stressing of the contagion matrix (stress scenario S.2), and under stressing of the frailty persistence (stress scenario S.3), are displayed in Figures 17-18, and 19-20, respectively.

[Insert Figure 17: Term structure of liquidation volatility when stressing the contagion matrix]

[Insert Figure 18: Term structure of liquidation overdispersion when stressing the contagion matrix]

[Insert Figure 19: Term structure of liquidation volatility when stressing the frailty persistence]

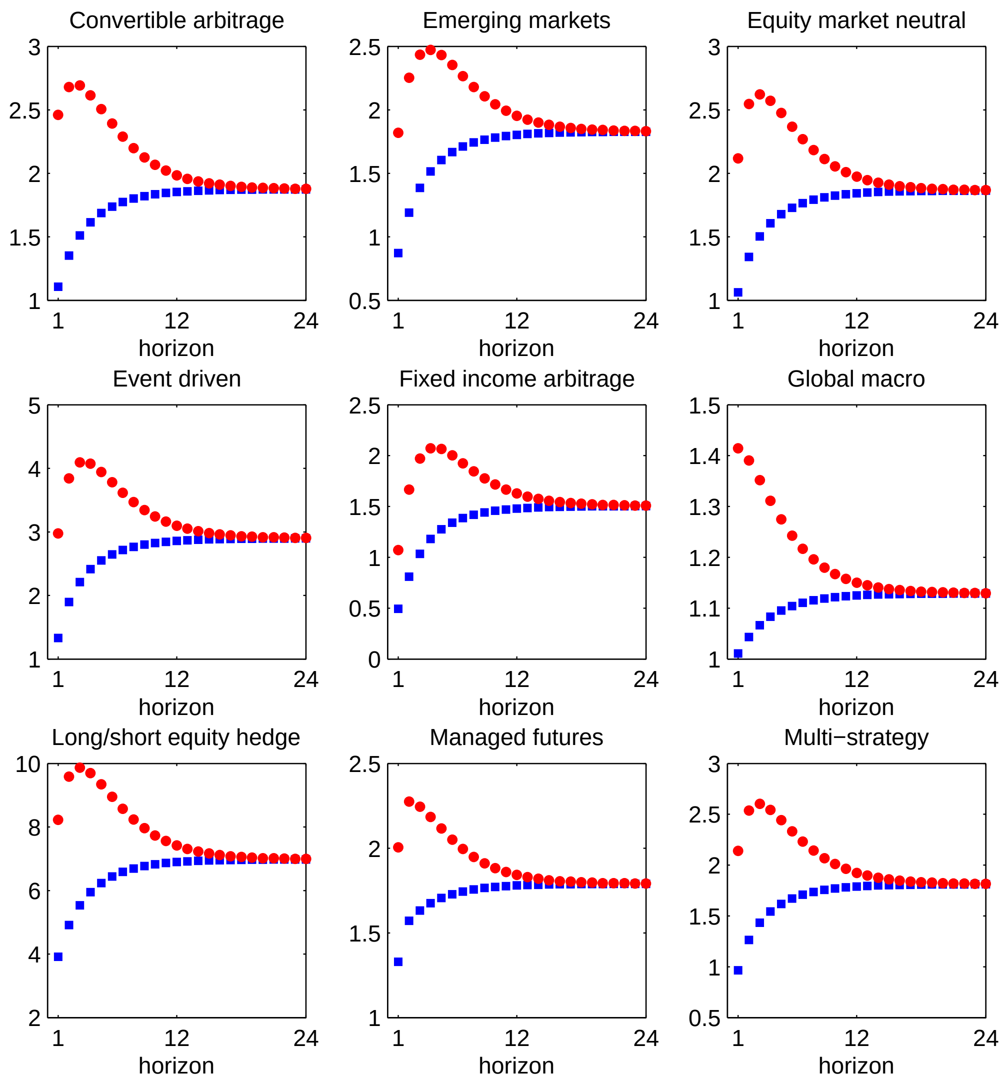
[Insert Figure 20: Term structure of liquidation overdispersion when stressing the frailty persistence]

In Figures 15 and 16, when the current value of the frailty is equal to the historical median $q_{0.5} = 0.520$, the term structures of volatility and overdispersion of the liquidation counts are upward sloping. As for the term structure of expected liquidation counts in Figure 12, this is due to the conditioning information, that includes historically low values of the liquidation counts at the current date, and a value of the frailty below the historical mean $E[F_t] = 1$. All management styles feature conditional overdispersion larger than $1$ for all horizons, which is due to the lagged liquidation counts (at horizons larger than one month) and the unobservable frailty. The shock on the factor value has a temporary effect on the term structures of volatility and overdispersion of the liquidation counts. Comparing Figures 15 and 16 with Figure 12, this effect features a larger time lag compared to the effect on the term structure of expected liquidation counts. In particular, the largest effects on the term structure of overdispersion are at about 3-6 months after the shock on the frailty. In Figures 17 and 18, the shock on the contagion matrix is a persistent shock, and the patterns of the effects on the term structures of volatility and overdispersion of the liquidation counts are rather similar as for the term structure of expected liquidation in Figure 13. Finally, in Figures 19 and 20, the effects of the shock to the frailty persistence on the term structures of volatility and overdispersion have different signs depending on the horizon. Indeed, in the medium/short term (up to $24$ months, say), the effect of the shock is to reduce the volatility and the overdispersion of the liquidation counts, while in the long term the effect is to increase volatility and overdispersion. This pattern is the result of the combination of two effects acting in opposite directions. In the short term, the dominating effect of the positive shock on the frailty persistence is to increase the likelihood of a future frailty value below the historical average. In the long term, the term structures of volatility and overdispersion converge to the unconditional values. From equation (b.3), the unconditional volatility and overdispersion of the liquidation counts are increasing functions of the frailty autocorrelation $\rho$, as long as the elements of sensitivity vector $b$ and contagion matrix $C$ are positive. Therefore, the increase of the frailty autocorrelation parameter corresponds to a persistent shock on the volatility and overdispersion of the liquidation counts. This explains the impact on their term structures in the long run. Such a long run impact is not observed in Figure 14, since the unconditional expectation of the liquidation counts is independent of the frailty persistence.

# References

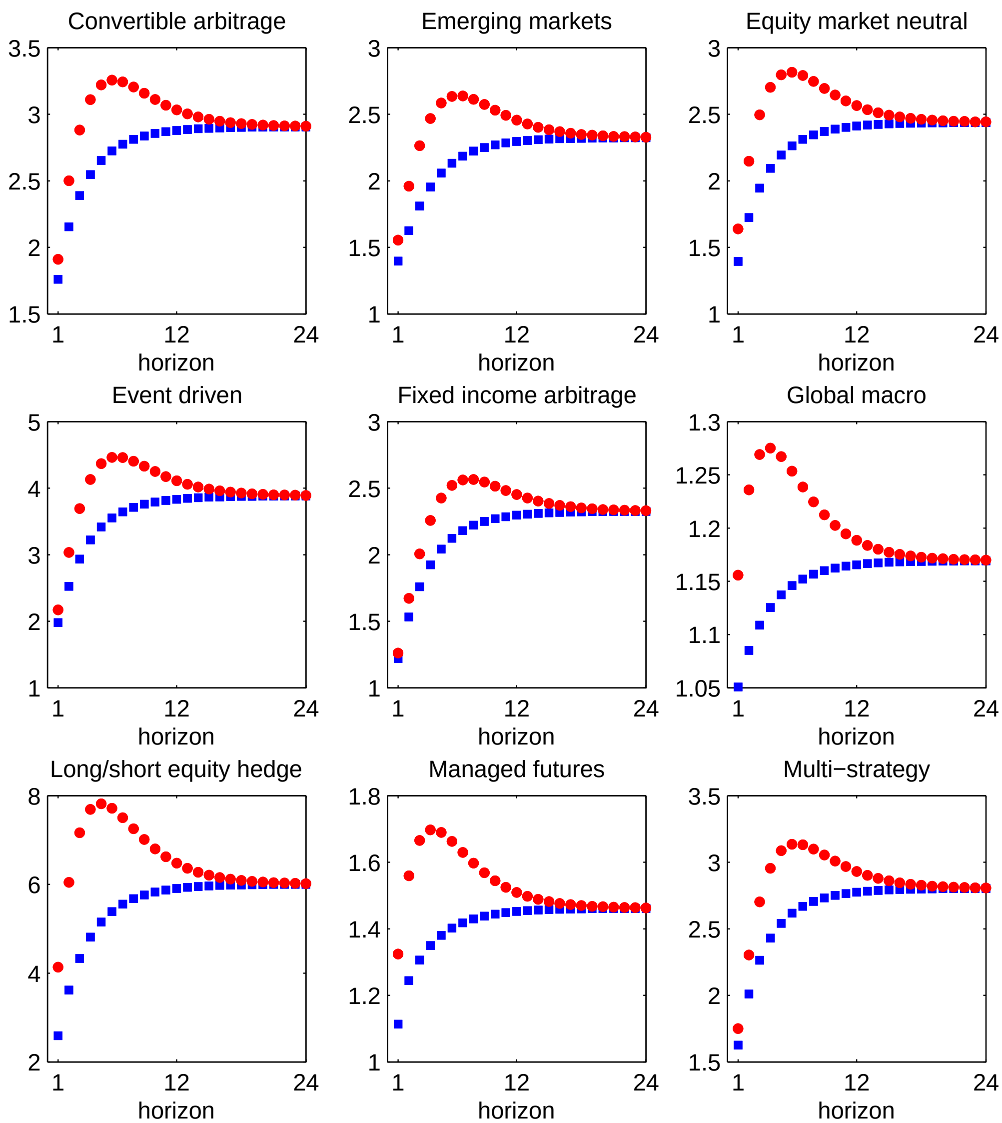
Magnus, J., and H., Neudecker (2007): *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons.

Figure 15: Term structure of liquidation volatility when stressing the current factor value



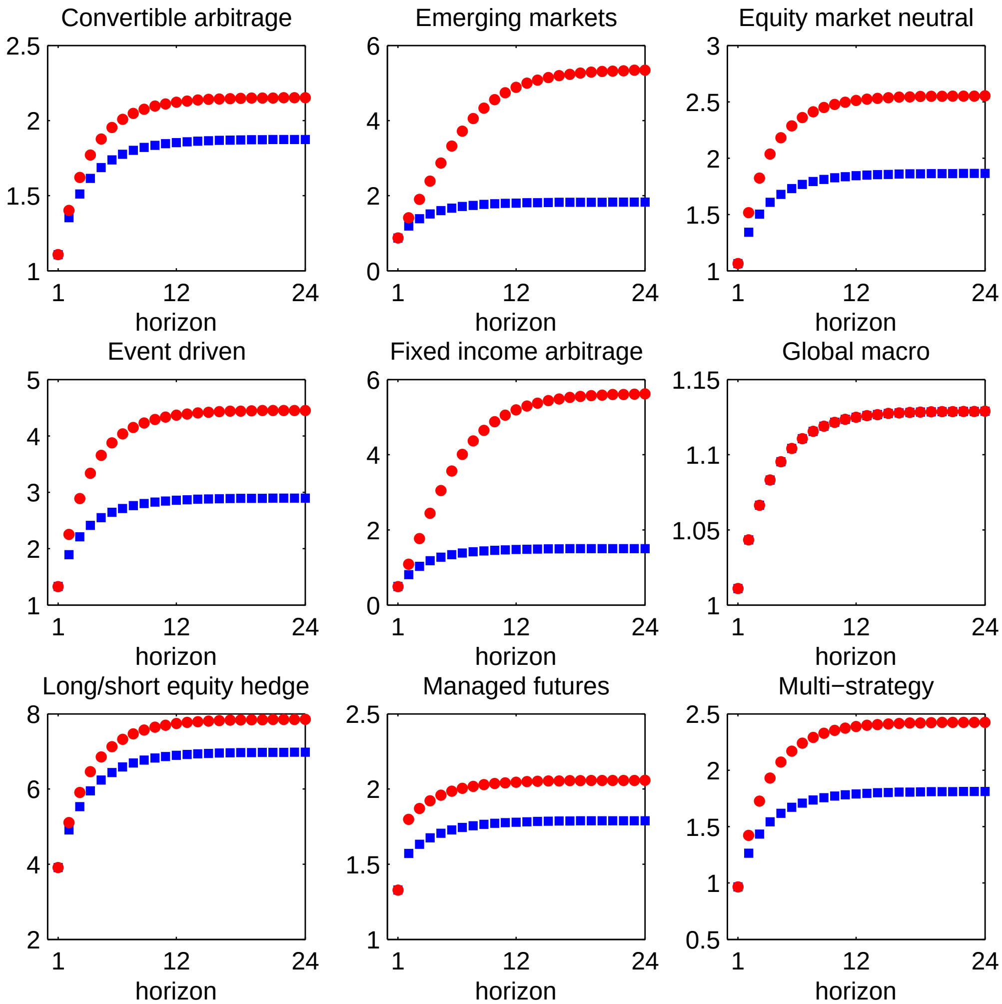

Term structure of conditional volatility $V[Y_{k,t+\tau}|Y_t, F_t]^{1/2}$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to conditioning sets with $F_t$ equal to the median and the $95\%$ quantile, respectively, of the stationary distribution of the frailty. The liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves. The model is the specification including frailty and contagion, with intensity parameters as in Tables 5 and 6, and frailty dynamic parameters $\delta = 0.59$ and $\rho = 0.74$, corresponding to the estimates of Section 3.4.

Figure 16: Term structure of liquidation dispersion when stressing the current factor value



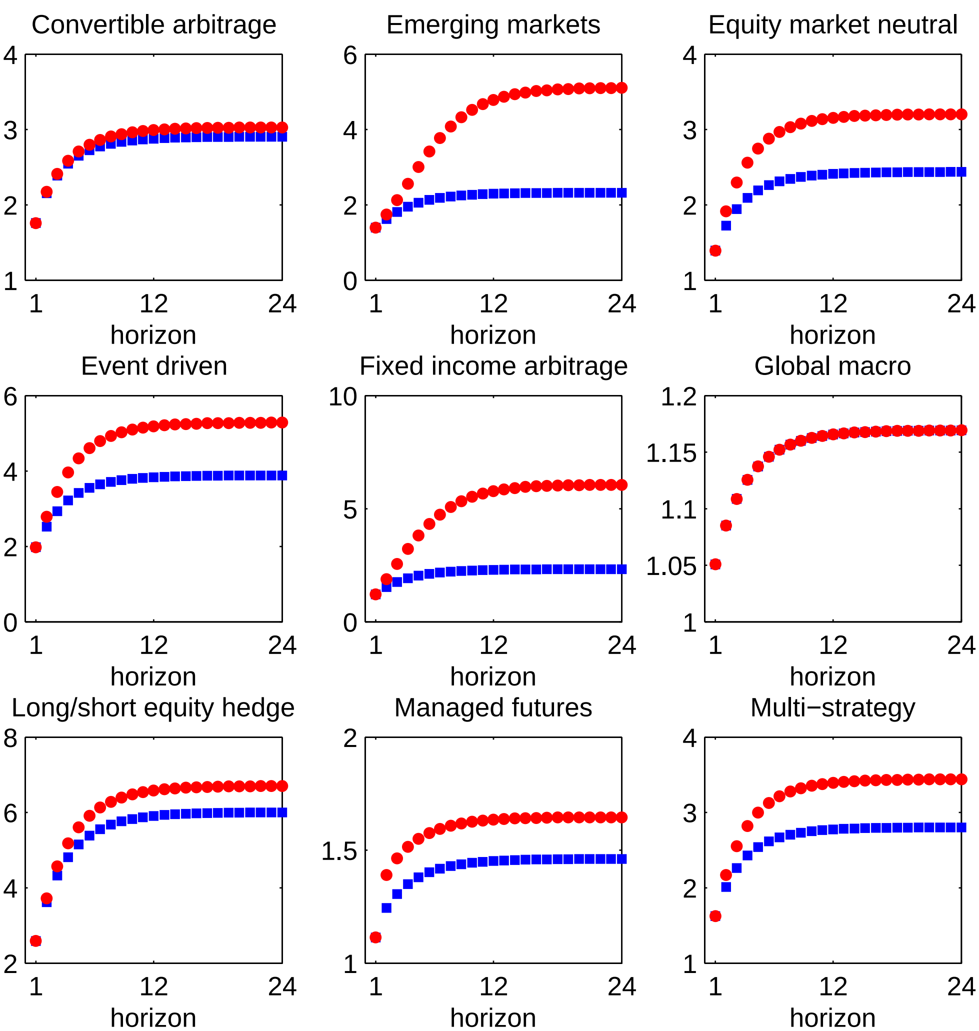

Term structure of conditional dispersion $V[Y_{k,t+\tau}|Y_t, F_t]/E[Y_{k,t+\tau}|Y_t, F_t]$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to conditioning sets with $F_t$ equal to the median and the $95\%$ quantile, respectively, of the stationary distribution of the frailty. The liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves. The model is the specification including frailty and contagion, with intensity parameters as in Tables 5 and 6, and frailty dynamic parameters $\delta = 0.59$ and $\rho = 0.74$, corresponding to the estimates of Section 3.4.

Figure 17: Term structure of liquidation volatility when stressing the contagion matrix



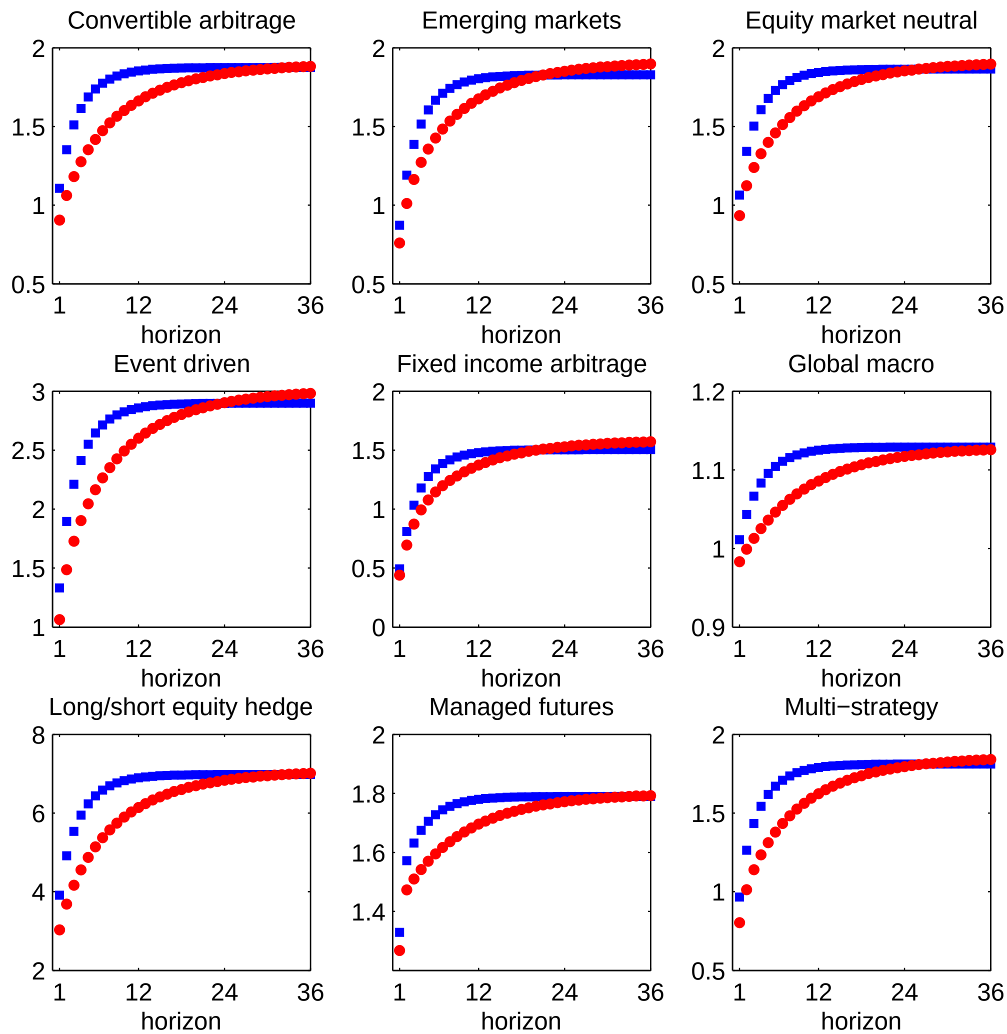

Term structure of conditional volatility $V[Y_{k,t+\tau}|Y_t, F_t]^{1/2}$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to models with contagion matrices $C^s = \hat{C}$ and $C^s = 2\hat{C}$, respectively, where $\hat{C}$ is the matrix of estimates in Table 6. The intercepts and frailty sensitivities are as in Table 5, and the frailty dynamic parameters are $\delta = 0.59$ and $\rho = 0.74$, corresponding to the estimates of Section 3.4. The factor value $F_t$ in the conditioning set corresponds to the median of the stationary distribution of the frailty, while the liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves.

Figure 18: Term structure of liquidation dispersion when stressing the contagion matrix



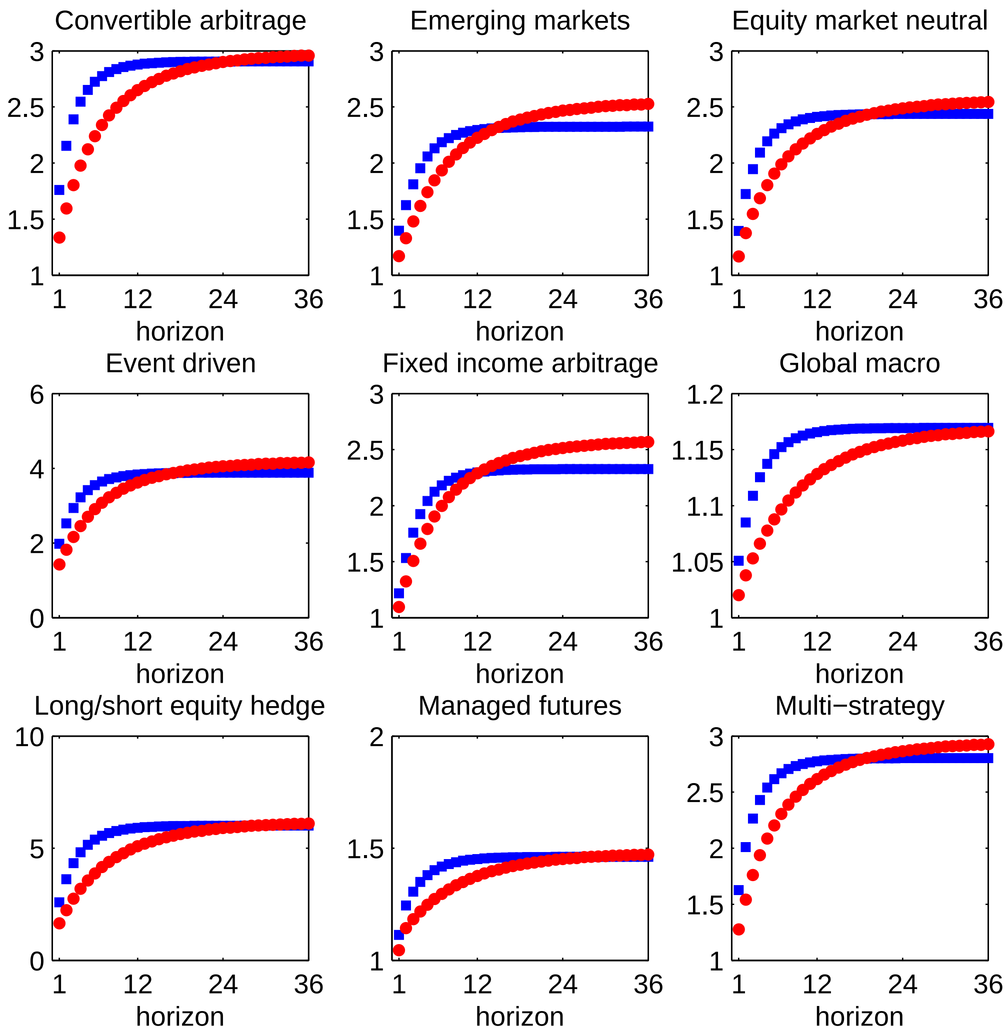

Term structure of conditional dispersion $V[Y_{k,t+\tau}|Y_t, F_t]/E[Y_{k,t+\tau}|Y_t, F_t]$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to models with contagion matrices $C^s = \hat{C}$ and $C^s = 2\hat{C}$, respectively, where $\hat{C}$ is the matrix of estimates in Table 6. The intercepts and frailty sensitivities are as in Table 5, and the frailty dynamic parameters are $\delta = 0.59$ and $\rho = 0.74$, corresponding to the estimates of Section 3.4. The factor value $F_t$ in the conditioning set corresponds to the median of the stationary distribution of the frailty, while the liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves.

Figure 19: Term structure of liquidation volatility when stressing the frailty persistence




Term structure of conditional volatility $V[Y_{k,t+\tau}|Y_t, F_t]^{1/2}$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to models with frailty autocorrelation $\rho^s = 0.74$ (corresponding to the estimate in Section 3.4) and $\rho^s = 0.90$, respectively. The intensity parameters are as in Tables 5 and 6, and the parameter characterizing the stationary distribution of the frailty is $\delta = 0.59$, corresponding to the estimate of Section 3.4. The factor value $F_t$ in the conditioning set corresponds to the median of the stationary distribution of the frailty, while the liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves.

Figure 20: Term structure of liquidation dispersion when stressing the frailty persistence




Term structure of conditional dispersion $V[Y_{k,t+\tau}|Y_t, F_t]/E[Y_{k,t+\tau}|Y_t, F_t]$ of liquidation counts for horizon $\tau = 1, 2, ..., 24$ months, by management style $k$. Squares and circles correspond to models with frailty autocorrelation $\rho^s = 0.74$ (corresponding to the estimate in Section 3.4) and $\rho^s = 0.90$, respectively. The intensity parameters are as in Tables 5 and 6, and the parameter characterizing the stationary distribution of the frailty is $\delta = 0.59$, corresponding to the estimate of Section 3.4. The factor value $F_t$ in the conditioning set corresponds to the median of the stationary distribution of the frailty, while the liquidation counts vector $Y_t$ in the conditioning set corresponds to the observations in June 2009 for both curves.