Positional Portfolio Management

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Abstract

In this paper we introduce and study positional portfolio management. In a positional allocation strategy, the manager maximizes an expected utility function written on the cross-sectional rank (position) of the portfolio return. The objective function reflects the goal of the manager to be well-ranked among his/her competitors. To implement positional allocation strategies, we specify a nonlinear unobservable factor model for the asset returns. The model disentangles the dynamics of the cross-sectional distribution of the returns and the dynamics of the ranks of the individual assets within the cross-sectional distribution. We estimate the model on a large set of stocks traded in the NYSE, AMEX and NASDAQ markets between 1990/1 and 2009/12, and implement the positional strategies for different investment universes. The positional strategies outperform standard momentum, reversal and mean-variance allocation strategies for most criteria. Moreover, the positional strategies outperform the equally weighted portfolio for criteria based on position.

JEL Codes: C38, C55, G11.

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1 Introduction

The management fees of portfolio managers should be designed to reconcile the objectives of these managers with the objectives of the investors. They depend on the asset under management for mutual funds, and also on the returns of the portfolio above some benchmark threshold, the so-called high-water mark, for hedge funds [Brown, Harlow, and Starks (1996), Aragon and Nanda (2012), Darolles and Gourieroux (2014)]. These designs might be not entirely satisfactory and induce spurious portfolio management. For instance, the effect of high-water mark can lead managers to take too risky short term positions and use a high leverage. Similarly, to increase his/her market share, that is the asset under management, the manager has to get better performance than his/her competitors. In this respect, the manager might be more interested in relative performance than in absolute performance, especially when the journals for investors write lead articles or even make their cover page on the ranking of funds. For instance, “in the Managed and Personal Investing section of the *Wall Street Journal Europe*, the Fund Scorecard provides the return of the top fifteen performers in a category” [Goriaev, Palomino, and Prat (2001)].

The traditional Finance theory assesses the quality of a portfolio management strategy by considering the expected (indirect) utility of the portfolio value, or of the portfolio return. A portfolio with 10% expected return is preferred to a portfolio with 8% expected return for a given level of risk. However, this preference ordering can be questioned if we account for the *context*, that is, for competing portfolio managements. Do we prefer a 10% return when the competing portfolio return is 20%, or a 8% return when the competing portfolio return is 5% ? Indeed, with 8% return the portfolio manager is number one, whereas he/she is not with 10% return. Economic theory uses the term *positional good* to “denote the good for which the link between context”, i.e., the behaviour of other economic agents, "and evaluation is the strongest", and the term *nonpositional good* to denote that for which the link is the weakest [Hirsch (1976), Frank (1991)]. Positional theory has proved useful to explain the escalation of expenditures in armaments, the race for technology in electronic financial markets [Biais, Foucault, and Moinas (2013)], the negative association between happiness measures and average neighbourhood income [Easterlin (1995), Frey and Stutzer (2002)], the sharp increase in the surface of newly constructed houses in the United States, the labour force participation of married women [Neumark and Postlewaite (1998)], and the demand for luxury goods [Frank (1999)]. The application of positional theory in Finance, which is the closest to the topic of this paper, is the competition for talented agents, especially for CEOs, fund managers, or traders in the finance sector [see e.g. Gabaix and Landier (2008), Thanassoulis (2012)]. Indeed, the fact that investors look for talented fund managers might explain the incentive for positioning introduced in the contracts for management fees, as well as the race of fund managers to be well ranked, i.e. the so-called fund tournament [Goriaev, Palomino, and Prat (2001), Goriaev, Nijman, and Werker (2005), Chen and Pennacchi (2009), Schwarz (2012)].

The aim of this paper is to introduce the positional concern in portfolio management. The positional portfolio management is based on the maximization of the expected utility of the future rank (or position) of the portfolio value, as opposed to the traditional portfolio management which focuses on the expected utility of the future portfolio value itself. The positional portfolio management leads to new types of allocations strategies, which we compare theoretically and empirically with traditional allocation strategies, such as mean-variance, momentum and contrarian (or reversal) strategies, as well as the naive $1/n$ portfolio. We measure the ability of positional strategies to yield portfolio returns that rank well cross-sectionally. A positional strategy diverts resources to be well ranked in the race among portfolio managers and might diminish the absolute performance compared to nonpositional strategies. In this respect, such a management does not necessarily act in the interest of investors.

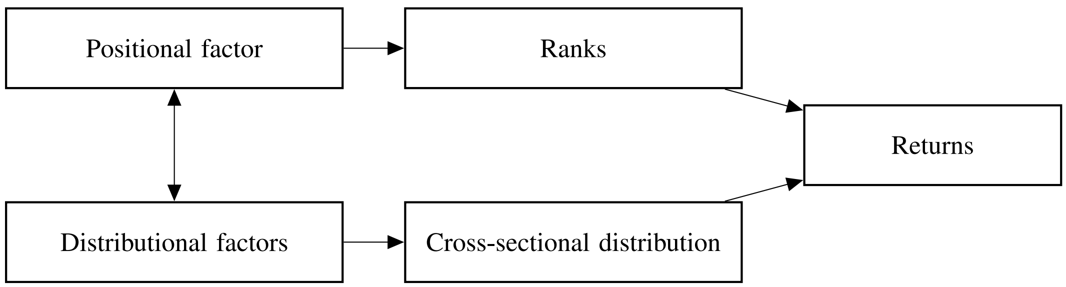
Therefore, one goal of our analysis is to measure the loss (or gain) of absolute performance due to a positional strategy.[1]

In Section 2, we introduce the notion of cross-sectional rank (position). This notion is used to define a positional portfolio management, and is at the core of the distinction of this management from the standard management based on the expected utility of future portfolio returns. A positional strategy can be interpreted as a standard strategy in which the utility function is replaced by a stochastic utility, which is function of the stochastic future cross-sectional distribution of returns. To implement the positional portfolio strategy we need an appropriate specification which disentangles the dynamics of the ranks from those of the cross-sectional distribution of returns. The model for the dynamics of ranks is introduced in Section 3. The Gaussian ranks follow a conditionally Gaussian autoregressive process, with the autoregressive coefficient accounting for positional persistence. The latter can depend on unobservable individual heterogeneities and stochastic dynamic factors. The dynamic model for the ranks is used in Section 4 to construct a first type of positional portfolio allocation strategies, which are compared with standard momentum and reversal strategies on a large panel of returns for stocks traded in the NYSE, AMEX and NASDAQ markets. The investment universe for these positional strategies consists of about 1000 stocks, which illustrates the big data aspect of our analysis. In Section 5 we complete the model by introducing an appropriate specification for the dynamics of the cross-sectional distribution of individual stock returns. The distribution is chosen in the Variance-Gamma family, with stochastic mean, variance, skewness and kurtosis driven by unobservable common factors, in order to accommodate time-varying higher-order moments of the cross-sectional returns distribution. The full vector of macro-factors driving positional persistence and the moments of the cross-sectional

[1] We do not discuss in this paper the positive or negative effects of such tournaments on the global social welfare, as possible misuse of resources on the negative side, and acceleration of innovation on the positive side.

distribution follows a vector autoregressive (VAR) process. The specifications for the dynamics of positions, cross-sectional distribution and underlying factors define the joint dynamics of returns. The complete dynamic model is summarized in Scheme 1.

Scheme 1: The model structure



This complete dynamic model is used in Section 6 to construct efficient positional portfolio allocation strategies. We compare the performance of the momentum and efficient positional strategies with the performance of traditional mean-variance, minimum-variance and $1/n$ strategies. We find that the positional strategies implemented out-of-sample outperform momentum and reversal strategies, as well as mean-variance and minimum-variance strategies in terms of average positional utility and Sharpe ratio. The performance of the positional strategies is similar to that of the equally-weighted portfolio according to these criteria, but the former outperform the latter in terms of probability to be well-ranked. Section 7 concludes. Technical proofs and a discussion of the Nash equilibrium of positional strategies are gathered in Appendices.

2 Positional portfolio management

2.1 Returns and positions

Let us consider a set of n risky assets $i = 1, \ldots, n$, which can be either stocks, or fund portfolios, and a riskfree asset with riskfree rate $r_{f,t}$. We denote by $y_{i,t}$ the return of risky asset i in period t, for $t = 1, ..., T$. At any given date, the observed returns can be used to define the ranks (or positions) of the assets. For this purpose, it is necessary to distinguish the ex-ante and ex-post notions of rank (or position). The ex-post ranks are simply obtained by ranking at any given date t the asset returns from the smallest one to the largest one, and then taking their positions in this ranking (divided by n). Formally, the ex-post ranks are defined as $\hat{u}^*_{i,t} = \hat{H}^*_t(y_{i,t})$, where $\hat{H}^*_t(\cdot)$ is the empirical cross-sectional (CS) cumulative distribution function (c.d.f.) of the returns at date t. In the ex-ante analysis, the empirical cross-sectional c.d.f. at the current date t is replaced by its theoretical analogue, denoted by $H^*_t(\cdot)$ (see Appendix 1). Then, the ex-ante ranks are given by $u^*_{i,t} = H^*_t(y_{i,t})$. The rank $u^*_{i,t}$ corresponds to the position of return $y_{i,t}$ ex-ante with respect to the observation of the other asset returns. The ex-ante ranks have a cross-sectional uniform distribution on the interval $[0, 1]$, whereas the ex-post ranks have the discrete empirical uniform distribution on $\{1/n, 2/n, ..., 1\}$.

Since the ranks are defined up to an increasing transformation, we can also introduce the ex-ante and ex-post Gaussian ranks. They are obtained from the corresponding uniform ranks by applying the quantile function of the standard normal distribution:

$$u_{i,t} = \Phi^{-1}(u^*_{i,t}) \qquad \text{and} \qquad \hat{u}_{i,t} = \Phi^{-1}(\hat{u}^*_{i,t}), \tag{2.1}$$

where Φ is the c.d.f. of the standard normal distribution. The ex-ante Gaussian ranks $u_{i,t}$ (resp. the ex-post Gaussian ranks $\hat{u}_{i,t}$) are standardized to ensure a cross-sectional standard normal distribution (resp. a cross-sectional distribution close to the standard normal one for large n). For instance, if asset i has ex-post rank $\hat{u}^*_{i,t} = 0.95$, there are 95% of assets in the sample with a smaller or equal return on time t, and 5% of assets with a larger return. The corresponding ex-post Gaussian rank is $\hat{u}_{i,t} = 1.64$, that is the 95% quantile of the standard normal distribution. If an asset i has ex-ante rank $u_{i,t} = 0.95$, there is a probability equal to 0.95 that the return at time t of any other asset is smaller or equal to the return of asset i. The ex-ante Gaussian ranks are related to the returns by the equation $u_{i,t} = H_t(y_{i,t})$, where H_t is the compound function $H_t = \Phi^{-1} \circ H^*_t$.

To illustrate the notions of ex-ante and ex-post cross-sectional distributions, we consider the subsample of all Center for Research in Security Prices (CRSP) common stocks [2] traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ, for which the monthly holding-period returns are available for the period ranging from January 1990 to December 2009. We exclude from the dataset the stocks for which monthly volume data are either missing, or equal to 0, at some months. We get a balanced panel for the returns of $n = 939$ companies, with $T = 240$ monthly observations. We compute the empirical cross-sectional distribution of returns $\hat{H}^*_t$ at the end of each month of the sample. The associated smoothed probability density functions are displayed in Figure 1.

[FIGURE 1: Time series of cross-sectional distributions of monthly CRSP stock returns.]

We deduce from these distributions the associated 5%, 25%, 50%, 75%, 95% empirical cross-sectional quantiles, which are time varying. The time series of these quantiles are displayed in Figure 2.

[FIGURE 2: Time series of quantiles of the CS distributions of monthly CRSP stock returns.]

[2] Common stocks are stocks with CRSP End of Period Share Code 10 and 11. Therefore, our sample does not include Certificates, American Depositary Receipts (ADR), Shares of Beneficial Interest (SBI), Units, Exchange-Traded Funds (ETF), Companies incorporated outside the U.S., Close-ended funds, and Real Estate Investment Trusts (REIT). Stock prices are denominated in US dollars. The CRSP dataset includes 15044 different common stocks listed in the NYSE, the AMEX and the NASDAQ, in the period from January 1990 to December 2009.

The empirical cross-sectional distributions are generally unimodal, with a mode close to the zero return. They vary over time, mainly in their concentration and tails. As expected, we observe in Figure 2 an endogenous clustering of these effects: the individual returns are more cross-sectionally concentrated at some periods of time, and less concentrated at some other ones.

In Figure 3 we consider an hypothetical riskfree asset with a constant monthly return 0.05 and provide the time series of its ex-post Gaussian ranks.

[FIGURE 3: Time series of ex-post Gaussian ranks associated with a constant monthly return of 0.05.]

This constant return is below the CS median in some months, and above the 95% CS quantile in other months. In Figure 3 these effects are reflected by the fact that the ex-post Gaussian rank is smaller than 0, or larger than 1.64, respectively, at some months.

2.2 Positional management strategies

Let us assume that the investor's information at date t, denoted by I_t, includes the current and past realizations of all asset returns: $I_t = (\underline{r_{f,t}}, \underline{y_t})$, where $\underline{y_t} = (y_t, y_{t-1}, \dots)$ and $y_t = (y_{1,t}, \dots, y_{n,t})'$. The standard (myopic) portfolio management summarizes the preferences of the investor by means of an increasing concave indirect utility function U written on the future portfolio value. The investor selects at time t the portfolio allocation which maximizes the expected utility of the future portfolio value. Let us consider a portfolio invested in both risky and riskfree assets and denote by γ the vector of dollar allocations in the risky assets, $w_r = \gamma' e$ the budget invested in the risky assets, and e the n-dimensional unit vector. Then, $\alpha = \gamma / w_r$ is the vector of relative allocations in the risky assets. By taking into account the budget constraint, the future portfolio value is equal to:

$$W_{t+1} = W_t(1 + r_{f,t}) + \gamma' \tilde{y}_{t+1} = W_t(1 + r_{f,t}) + w_r \alpha' \tilde{y}_{t+1},$$

where W_t is the portfolio value at date t and $\tilde{y}_{t+1} = y_{t+1} - r_{f,t} e$ is the vector of excess returns. The optimization problem provides the optimal allocations $\hat{\gamma}_t$ by:

$$\hat{\gamma}_t = \arg\max_{\gamma} E_t \left(U[W_t(1 + r_{f,t}) + \gamma' \tilde{y}_{t+1}] \right), \tag{2.2}$$

where $E_t(\cdot) = E(\cdot | I_t)$ is the conditional expectation given the available information at time t, and the allocation $\hat{\gamma}_t$ can depend on this information. The optimal values $\hat{\gamma}_t$, $\hat{w}_{r,t} = \hat{\gamma}_t' e$ and $\hat{\alpha}_t = \hat{\gamma}_t / \hat{w}_{r,t}$ are also solutions of the two equivalent constrained optimization problems:

$$\begin{aligned} \hat{\gamma}_t &= \arg\max_{\gamma} E_t \left(U[W_t(1 + r_{f,t}) + \gamma' \tilde{y}_{t+1}] \right), \\ &\quad \text{s.t. } \gamma' e = \hat{w}_{r,t}, \end{aligned}$$

and:

$$\begin{aligned} \hat{\alpha}_t &= \arg\max_{\alpha} E_t \left(U[W_t(1 + r_{f,t}) + \hat{w}_{r,t} \alpha' \tilde{y}_{t+1}] \right), \\ &\quad \text{s.t. } \alpha' e = 1. \end{aligned} \tag{2.3}$$

Thus, the optimization can be splitted into two parts. In a first step we consider the optimal allocation of the total budget between the riskfree asset and the set of risky assets, that is $W_t - \hat{w}_{r,t}$ and $\hat{w}_{r,t}$. Then, the budget $\hat{w}_{r,t}$ is allocated between risky assets. For a CARA indirect utility function and conditionally Gaussian returns, we get the standard mean-variance efficient allocation [see e.g. Ingersoll (1987), p. 98]. In this case the quantity $\hat{w}_{r,t}$ depends on the risk aversion and on the conditional distribution of excess returns, but not on the initial portfolio value W_t. The relative allocations vector $\hat{\alpha}_t$ depends on the conditional distribution of excess returns only:

$$\hat{\alpha}_t = \frac{1}{e' \left[V_t(\tilde{y}_{t+1})\right]^{-1} E_t(\tilde{y}_{t+1})} \cdot \left[V_t(\tilde{y}_{t+1})\right]^{-1} E_t(\tilde{y}_{t+1}).$$

The objective of a fund manager could be, for instance, to provide a high portfolio (excess) return, or perhaps to provide a better (excess) return than his competitors. In the latter case, he can prefer to be in the "top ten", whatever the return levels are. Such a positional strategy can be developed for the whole portfolio including both riskfree and risky assets, or only for the risky part of the portfolio once the budgets for the riskfree and risky parts of the portfolio have been fixed. We follow the second approach, that is, we derive the optimal positional allocations vector γ subject to the constraint $\gamma' e = w_r$, for w_r given. We keep the same definition of the ranks as in Section 2.1, that is, we compare the position of portfolios with the positions of each individual stock. Thus, in a first step we consider as exogenous competitors portfolios invested in single assets. We show in Appendix 2 *i)* that the optimal positional strategy is $\gamma_t^* = w_r \alpha_t^*$, where:

$$\begin{aligned} \alpha_t^* &= \underset{\alpha:\, \alpha' e = 1}{\arg\max}\, E_t \left[\mathscr{U}\left(H_{t+1}(\alpha' y_{t+1})\right)\right] && (2.4) \\ &= \underset{\alpha:\, \alpha' e = 1}{\arg\max}\, E_t \left[\mathscr{U}\left(H_{t+1}\left(\sum_{i=1}^{n} \alpha_i H_{t+1}^{-1}(u_{i,t+1})\right)\right)\right], && (2.5) \end{aligned}$$

where $\mathscr{U}(\cdot)$ is a utility function written on the Gaussian rank $H_{t+1}(\alpha' y_{t+1})$ of the future return $\alpha' y_{t+1}$ of the risky part of the portfolio. Equation (2.5) leads to three remarks. First, the optimal positional relative allocations vector α_t^* is independent of w_r, i.e., it can be computed for a risky portfolio of unitary value 1. The reason is that a positional strategy is not interested in the levels of the portfolio values, but only on their comparison. Second, the ranks are computed on the returns. Indeed, the ranks computed on the returns, or on the excess returns, are the same. Third, in equation (2.5) the future portfolio rank $H_{t+1}\left(\sum_{i=1}^{n}\alpha_i H_{t+1}^{-1}(u_{i,t+1})\right)$ is a nonlinear aggregate of the individual future ranks [see Appendix 2 *ii)*]. The nonlinear aggregation scheme involves the stochastic future cross-sectional distribution of returns H_{t+1}^* via function $H_{t+1} = \Phi^{-1} \circ H_{t+1}^*$.

By comparing equations (2.3) and (2.4), we note that the positional utility function $\mathscr{U}$ is different from the rescaled indirect utility function U_t, with $U_t(r) = U[W_t(1 + r_{f,t}) + \hat{w}_{r,t}r]$, written on the portfolio excess return $r = \alpha'\tilde{y}_{t+1}$ of the risky part of the portfolio. In particular, the argument of the rescaled indirect utility function U_t admits the unit: $ at time $t+1$ over $ at time t, while the argument of the positional utility function $\mathscr{U}$ is dimensionless. A positional strategy replaces the increasing and concave rescaled utility function U_t by an endogenous stochastic utility function $U_{t+1} = \mathscr{U} \circ H_{t+1}$, which is strictly increasing, but non-concave in general. The positional portfolio management depends on the choice of the positional utility function $\mathscr{U}$, but also on the selected definition of ranks, that can be uniform or Gaussian, and on the universe of stocks used to compute these ranks. Moreover, the optimal allocation α_t^* of the fund manager is defined by considering in a first step the function H_{t+1} as exogenous. In particular, we have chosen the return distribution of portfolios invested in single assets as such exogenous benchmark. When the rank is computed with respect to the performance of other managed portfolios, the future "portfolio returns" distribution has also to account for the possible reactions of the other fund managers, who also want to be in the "top ten". In this case, all portfolios associated with the funds would be optimized jointly. Thus, the performance of the fund is seen as a public good [see e.g. Hirsch (1976), Frank (1991)]. In other words, if we considered the positional equilibrium condition as the analogue of the standard CAPM, the equilibrium would be with respect to the prices, information set and also to the cross-sectional distribution H_{t+1}. In Appendix 2 *iii)* we describe the Nash equilibrium for the positional allocation problem of fund managers in such a complex framework. The objective function in equation (2.4) would involve the behaviours of the other fund managers. Therefore, our analysis is related to the literature on social interactions [see e.g. Davezies, D'Haultfoeuille, and Fougère (2009) and Blume, Brock, Durlauf, and Jayaraman (2013)], especially the part of this literature interested in strategic complementarities in production [Calvo-Armengol, Patacchini, and Zenou (2009)]. However, it differs from this literature because of the more sophisticated objective function which is considered [3]. In particular, at the equilibrium we do not get bilateral effects only, i.e. peer effects only. In our framework the individual decision involves in a complicated way the complete distribution of other managers' decisions. The extension to an endogenous benchmark H_{t+1} is not considered further in the main body of the paper.

The preferences based on expected positional utility satisfy some axioms of the expected utility theory introduced by von Neumann and Morgenstern (1944), but not all of them. For instance, the expected positional utility is a linear function of the probability of the future state including in our case the returns of all assets. However, the compatibility with the second-order stochastic dominance for the portfolio returns is clearly not satisfied, since the preferences also involve the distributions of the other stock returns.[4]

In order to implement the positional strategies defined in (2.5) and to compare them with the standard allocation strategies based on the expected utility of future portfolio values, we need an appropriate dynamic model for both the rank processes and the transformed cross-sectional distribution

[3]In the standard social interaction models, the individual utility functions are quadratic functions of the individual actions (i.e. the portfolio allocations). They depend on these individual actions by summaries interpretable as cross-sectional portfolio values up to a change of probability. This leads to linear models for the Bayesian-Nash equilibrium strategies.

[4]In this respect our approach differs from the theory of anticipated utility, also called rank dependent expected utility theory. This theory introduced by Quiggin (1982) [see also Kahneman and Tversky (1992) for the extension to prospect theory] is using some ranks to overweigh rare extreme events. In this approach, the rank would be computed as the rank of the portfolio value across states of nature, that is, with respect to the (conditional) distribution of the portfolio return. In our framework the rank is computed with respect to the cross-sectional distribution of the stock returns. Thus, the benchmark distribution used to compute the rank varies with the state of nature.

H_t linking the returns and the ranks. An illustrative example is discussed below and is extended in the next sections to accommodate the empirical features of the return processes.

2.3 A toy-model of cross-sectional Gaussian returns

Let us consider a simple joint dynamic model for returns and Gaussian ranks, in which the returns are *cross-sectionally* Gaussian and the ranks are serially persistent. We use this toy-model to provide a simple intuition for positional portfolio allocation, and will extend it for empirical analysis.

The model is defined in two steps, by specifying first the dynamics of the Gaussian ranks and then the link between the individual asset returns and their ranks. The ex-ante Gaussian ranks $u_{i,t}$ are assumed such that:

$$u_{i,t} = \rho u_{i,t-1} + \sqrt{1-\rho^2}\varepsilon_{i,t}, \tag{2.6}$$

where the idiosyncratic disturbance terms $\varepsilon_{i,t}$ are independent and identically distributed (i.i.d.) standard normal variables. The autoregressive coefficient ρ has a modulus smaller than 1 in order to ensure the stationarity of the process of Gaussian ranks. The unconditional distribution of $u_{i,t}$ coincides with the theoretical cross-sectional distribution and is standard normal. When coefficient ρ increases, the position of any asset features more serial persistence.

Suppose that the returns are defined from the Gaussian ranks by an affine stochastic transformation:

$$y_{i,t} = \sigma_t u_{i,t} + \mu_t, \tag{2.7}$$

where the scale and drift coefficients define the macro-dynamic factor $F_t = (\mu_t, \sigma_t)'$. The scale σ_t, that is the cross-sectional standard deviation, is a strictly positive process. Then, the cross-sectional return distribution at date t is Gaussian $N(\mu_t, \sigma_t^2)$. The function H_t mapping returns into Gaussian ranks is given by $H_t(y) = (y - \mu_t)/\sigma_t$. It simply consists in cross-sectionally demeaning and standardizing the returns. The individual return processes are not Gaussian, since they feature stochastic mean and variance due to factors μ_t and σ_t.

Let us now consider a portfolio invested in both risky and riskfree assets, with relative risky allocation vector α. The future return of the risky part of the portfolio is given by:

$$\alpha' y_{t+1} = \sigma_{t+1}\alpha' u_{t+1} + \mu_{t+1},$$

since $\alpha' e = 1$, and the corresponding excess return is:

$$\alpha' \tilde{y}_{t+1} = \sigma_{t+1}\alpha' u_{t+1} + \mu_{t+1} - r_{f,t}.$$

The future position of return $\alpha' y_{t+1}$ is:

$$H_{t+1}(\alpha' y_{t+1}) = \frac{(\sigma_{t+1}\alpha' u_{t+1} + \mu_{t+1}) - \mu_{t+1}}{\sigma_{t+1}} = \alpha' u_{t+1}. \tag{2.8}$$

Thus, the position of the future return of the risky part of the portfolio is a linear combination of the Gaussian ranks of the individual risky assets, with weights equal to the relative risky allocations α. This property is a consequence of the linearity of the (transformed) quantile function $H_{t+1}(\cdot)$, that is, of the Gaussian assumption for the CS distribution, and holds for any dynamics of the ranks. By taking into account the dynamics (2.6) of the Gaussian ranks, we get:

$$\alpha' \tilde{y}_{t+1} = \sigma_{t+1}\rho\alpha' u_t + \sigma_{t+1}\sqrt{1-\rho^2}\alpha' \varepsilon_{t+1} + \mu_{t+1} - r_{f,t}, \tag{2.9}$$

and:

$$H_{t+1}(\alpha' y_{t+1}) = \rho\alpha' u_t + \sqrt{1-\rho^2}\alpha' \varepsilon_{t+1}. \tag{2.10}$$

In the standard approach to portfolio management, we assume a CARA indirect utility function

$U(W;A) = -\exp(-AW)$ written on the portfolio value, where $A > 0$ is the absolute risk aversion of the investor. From (2.3) and (2.9) the expected utility is:

$$
\begin{aligned}
& -E[\exp(-AW_t(1+r_{f,t}) - Aw_r\alpha'\tilde{y}_{t+1})|\underline{F_t}, \underline{y_t}] \\
= & -\exp(-AW_t(1+r_{f,t}))E\left\{E[\exp(-Aw_r\alpha'\tilde{y}_{t+1})|\underline{F_{t+1}}, \underline{y_t}]|\underline{F_t}, \underline{y_t}\right\} \\
= & -\exp(-A(W_t+(W_t-w_r)r_{f,t}))E\left\{\exp(-Aw_r\sigma_{t+1}\rho\alpha' u_t - Aw_r\mu_{t+1} + \frac{A^2}{2}w_r^2\sigma_{t+1}^2(1-\rho^2)\alpha'\alpha)|\underline{F_t}, \underline{y_t}\right\}.
\end{aligned}
$$

The optimal portfolio is obtained by maximizing the above expected utility with respect to w_r and α subject to $\alpha' e = 1$. The optimal allocation depends on the joint dynamics of the cross-sectional mean and cross-sectional variance. If these dynamics are Markovian and exogenous with respect to the ranks, the optimal allocation depends on the current factor values (μ_t, σ_t) and ranks vector u_t. The allocations $\hat{\gamma}_t$ and $\hat{\alpha}_t$ in the risky assets are independent of the initial portfolio value W_t.

In the positional approach, we assume a CARA utility function $\mathscr{U}(v;\mathscr{A}) = -\exp(-\mathscr{A}v)$ written on the Gaussian rank of the future return of the risky part of the portfolio, with a positional risk aversion parameter $\mathscr{A} > 0$. By using equation (2.10), the expected positional utility is:

$$
-E[\exp(-\mathscr{A}H_{t+1}(\alpha' y_{t+1}))|\underline{F_t}, \underline{y_t}] = -\exp\left(-\mathscr{A}\rho\alpha' u_t + \frac{\mathscr{A}^2}{2}(1-\rho^2)\alpha'\alpha\right).
$$

The expected positional utility is independent of the factor values at time t and depends on the returns histories by means of the current positions vector u_t only. The optimal positional portfolio allocation is derived by maximizing $\mathscr{A}\rho\alpha' u_t - \dfrac{\mathscr{A}^2}{2}(1-\rho^2)\alpha'\alpha$ with respect to vector α subject to the budget constraint $\alpha' e = 1$. We get the optimal relative positional allocation in the risky assets:

$$
\alpha_t^* = \frac{1}{n}e + \frac{1}{\mathscr{A}}\frac{\rho}{1-\rho^2}(u_t - \bar{u}_t e), \tag{2.11}
$$

where $\bar{u}_t = u_t' e/n$ denotes the cross-sectional average of the Gaussian ranks at date t. This cross-sectional average tends to 0, which is the mean of the standard normal distribution, when the number of assets n tends to infinity. The optimal relative positional allocation α_t^* is a linear combination of two popular portfolios. The first one is the equally weighted portfolio, with weight $1/n$ in each asset [see e.g. DeMiguel, Garlappi, and Uppal (2009) and Beleznay, Markov, and Panchekha (2012)]. We see from (2.10) that this portfolio minimizes the conditional variance of future portfolio rank. For large n, the $1/n$ portfolio ensures the risk free median rank, but its return is still risky due to the effect of macro-factors. The second portfolio is an arbitrage portfolio (zero-cost portfolio) with dynamic allocations proportional to the current ranks of the assets in deviation from their cross-sectional average. The weight of the arbitrage portfolio in the relative risky allocation α_t^* is increasing with respect to the persistence ρ of the ranks, and decreasing with respect to the positional risk aversion coefficient $\mathscr{A}$ of the investor. The optimal positional allocation α_t^* deviates from the $1/n$ portfolio by overweighting the assets with larger (resp. smaller) current ranks, when the persistence parameter is positive (resp. negative)[5]. Thus, in this example the optimal positional allocation strategy combines the $1/n$ portfolio with momentum (resp. reversal) kind of strategies. The term $\rho(u_t - \bar{u}_t e)$ in equation (2.11) is equal to the vector of expected future ranks in deviation from their cross-sectional average. Thus, we can also interpret the arbitrage portfolio in (2.11) as a portfolio investing long in assets with large expected future rank and short in assets with small expected future rank, irrespective of the sign of the persistence parameter. This interpretation applies to more general specifications of the individual ranks dynamics, as shown in the next section.

[5]Even if the model of the example is symmetric in the individual assets, the portfolio allocation is not symmetric in the assets, since they have different returns, and then ranks, at date t.

3 The dynamics of positions

This section extends the toy dynamic model of positions in Section 2.3 to accommodate relevant empirical features. The main issue is that in our sample positional persistence varies across stocks and time [see Appendix 3 for evidence based on an Analysis of Variance (ANOVA)]. Therefore, we let the positional persistence depend on both stock-specific random effects and stochastic common dynamic factors.

3.1 Model specification

The joint dynamics of the individual Gaussian rank processes $(u_{i,t})$ is now specified as:

$$u_{i,t} = \rho_{i,t} u_{i,t-1} + \sqrt{1-\rho_{i,t}^2}\varepsilon_{i,t}, \tag{3.1}$$

$$\rho_{i,t} = \Psi(\beta_i + \gamma_i F_{p,t}), \tag{3.2}$$

where *i)* the idiosyncratic shocks $(\varepsilon_{i,t})$, the individual random effects $\delta_i = (\beta_i, \gamma_i)'$, and the macro-factor $F_{p,t}$ are mutually independent, *ii)* the shocks $(\varepsilon_{i,t})$ are standard Gaussian white noise processes independent across assets, and *iii)* the individual random effects δ_i are i.i.d. across assets. In equation (3.1) we assume that the Gaussian rank process $(u_{i,t})$ of any stock follows a conditionally Gaussian first-order Auto-Regressive [$AR(1)$] model. The autoregressive coefficient $\rho_{i,t}$ characterizes the positional persistence of stock i between months $t-1$ and t. The dependence of the autoregressive coefficient $\rho_{i,t}$ on the macro-factor and the individual effects is specified in equation (3.2). The single stochastic factor $F_{p,t}$ drives the positional persistence over time, that is, it is a positional macro-factor, whose interpretation has to be discussed jointly with the interpretation of the distributional macro-factors driving the cross-sectional return distribution (see Section 5). The individual effects β_i and γ_i

introduce heterogeneity across stocks in the long run average positional persistence and in the sensitivity to the positional persistence factor, respectively. We select function $\Psi(s) = (e^{2s} - 1)/(e^{2s} + 1)$, for $s \in \mathbb{R}$, to guarantee an autoregressive coefficient $\rho_{i,t}$ between -1 and 1 and to get a one-to-one increasing relationship between $\rho_{i,t}$ and the positional persistence score $\beta_i + \gamma_i F_{p,t}$. Since $\Psi(s) \approx s$ for an argument s close to 0, the positional persistence $\rho_{i,t}$ is approximately equal to the score $\beta_i + \gamma_i F_{p,t}$, when the latter is small in absolute value. The model in equations (3.1)-(3.2) extends specification (2.6) to individual and time dependent positional persistence. The joint process of individual Gaussian ranks defined in equations (3.1)-(3.2) satisfies the constraint of a standard Gaussian CS distribution [see Appendix 4, Subsections *i)* and *ii)*].

As usual in latent factor models, the factor values and the factor loadings are identifiable up to a one-to-one linear (affine) transformation. Indeed, systems $(F_{p,t}, \beta_i, \gamma_i)$ and $(cF_{p,t} + d, \beta_i - d/c, \gamma_i/c)$ are observationally equivalent, for any values of constants c and d, with $c \neq 0$. Therefore, without loss of generality, we assume:

$$E(F_{p,t}) = 0, \qquad E(F_{p,t}^2) = 1, \tag{3.3}$$

for identification purpose. Thus, for an asset i with small β_i and γ_i, the historical mean and variance of the positional persitence are approximately β_i and γ_i^2, respectively.

3.2 Model estimation

Let us now estimate the model of ranks dynamics on the dataset of $n = 939$ CRSP stocks described in Section 2.1.

i) Estimation procedure

We estimate the values of the positional persistence factor $F_{p,t}$ at all months t, and heterogeneities

β_i and γ_i for all stocks i, by maximizing the Gaussian conditional log-likelihood function of rank processes $(u_{i,t})$ after replacing the unobservable ex-ante Gaussian rank $u_{i,t}$ with the empirical ex-post Gaussian rank $\hat{u}_{i,t}$ defined in Section 2. Indeed, the ex-post and ex-ante ranks are close, when the cross-sectional size n is large[6]. We treat factor values and individual heterogeneities as unknown parameters. The fixed effects estimators $\hat{F}_{p,t}$ of the factor values, for $t = 1, ..., T$, and $\hat{\beta}_i$, $\hat{\gamma}_i$ of the heterogeneities, for $i = 1, ..., n$, are obtained from the maximization problem:

$$\max_{\substack{F_{p,t},\, t=1,...,T \\ \beta_i,\gamma_i,\, i=1,...,n}} \sum_{t=1}^{T}\sum_{i=1}^{n}\left\{-\frac{1}{2}\log(1-\rho_{i,t}^2) - \frac{(\hat{u}_{i,t}-\rho_{i,t}\hat{u}_{i,t-1})^2}{2(1-\rho_{i,t}^2)}\right\}, \tag{3.4}$$

where $\rho_{i,t} = \Psi(\beta_i + \gamma_i F_{p,t})$, subject to the constraints:

$$\frac{1}{T}\sum_{t=1}^{T} F_{p,t} = 0, \qquad \frac{1}{T}\sum_{t=1}^{T} F_{p,t}^2 = 1. \tag{3.5}$$

The constraints (3.5) are the empirical analogues of the identification conditions (3.3). In Appendix 4 *iii)*, we provide a sequential updating algorithm for the iterative computation of the estimates that are solutions of the constrained maximization problem (3.4)-(3.5). This sequential updating algorithm avoids the inversion of matrices of dimensions (n, n) or (T, T) corresponding to the parameter dimension. Thus, it has a small degree of numerical complexity appropriate in our big data framework.

[6]This is a consequence of the Law of Large Numbers (LLN) applied cross-sectionally conditionally on the path of macro-factors.

ii) Empirical results

We provide in Figure 4 the time series of factor estimates $\hat{F}_{p,t}$. The estimated serial autocorrelations are not significant. Thus, we will assume that the factor values $F_{p,t}$ are independent and identically distributed over time. This assumption implies the independence across time of common shocks to positional persistence, but not the absence of positional persistence itself.

[FIGURE 4 : Time series of positional factor estimates $\hat{F}_{p,t}$.]

[FIGURE 5 : Positional factor vs. CRSP EW index returns.]

Figure 5 shows a negative association between the estimated positional factor values and the monthly returns of the equally weighted (EW) CRSP index returns, at least for the months with negative EW CRSP index returns. This finding is similar to Figure 4 in Moskowitz, Ooi, and Pedersen (2012) who report a U-shape relationship between their momentum strategy and the S&P 500 index returns.

Let us now consider the estimated heterogeneity parameters $\hat{\beta}_i$ and $\hat{\gamma}_i$. Their marginal distributions are displayed in Figure 6, and some insight on their joint distribution is given by the scatterplot in Figure 7. The marginal distributions are unimodal and the values of $\hat{\beta}_i$ and $\hat{\gamma}_i$ in the support have the same order of magnitude. The marginal distribution of the $\hat{\beta}_i$ is close to a Gaussian distribution, while the marginal distribution of $\hat{\gamma}_i$ features right skewness. Thus, for large positive (resp. negative) values of positional factor $F_{p,t}$, we expect a large proportion of stocks with large positive (resp. negative) positional persistence. Figure 7 shows that the nonparametric regression of $\hat{\gamma}_i$ on $\hat{\beta}_i$ is almost linear. We observe a significant positive slope in this regression. Hence, the stocks which feature more positional persistence on average, also feature more time variation in this positional persistence.[7]

[7] We find some association between the individual effects (β_i, γ_i) and the industrial sectors. For instance the sector with the largest average value of sensitivity γ_i to the positional factor is Energy, and the one with the smallest average value is Healthcare, Medical Equipment, and Drugs. A standard t-test rejects the null hypothesis that the mean values of the distributions of the γ_i in the two sectors are the same.

[FIGURE 6 : Histograms of estimated individual effects.]

[FIGURE 7 : Scatterplot of $\hat{\gamma}_i$ vs. $\hat{\beta}_i$.]

The effect of the heterogeneity parameters on positional persistence is rather complex, since it involves the distribution of individual effects β_i and γ_i including their dependence, the level of the factor $F_{p,t}$, and passes through the nonlinear transformation Ψ. Figure 8 displays the distribution of the positional persistence for different factor levels.

[FIGURE 8 : Histograms of positional persistence $\hat{\rho}_{i,t}$ as function of $F_{p,t}$.]

When the positional factor value is negative (resp. positive), we observe a negative (resp. positive) average value of the positional persistence. When the factor $F_{p,t}$ gets larger in absolute value, the main effect comes from the γ_i distribution and the dispersion of the distribution of positional persistence increases. When the factor is close to 0, corresponding to the median of its distribution, we observe mainly the distribution of sensitivities β_i. Overall Figures 6 - 8 show that the estimated model accommodates for some stocks featuring momentum and others featuring reversal at a given date. Moreover, a given stock might feature momentum at some dates, and reversal at other dates, depending on the value of the positional factor $F_{p,t}$.

3.3 Efficient positional allocation with Gaussian CS distribution

In this section we derive the optimal positional allocation when the individual positions follow the autoregressive model (3.1)-(3.2) with stochastic persistence, and the CS distributions of returns are in the Gaussian family with stochastic mean and variance as in (2.7). Let us consider the CARA positional utility function $\mathscr{U}(v) = -\exp(-\mathscr{A}v)$, with $\mathscr{A} > 0$. The optimal positional allocation α_t^*

defined in (2.4) is such that [see Appendix 4 *iv)*]:

$$\alpha_{i,t}^* = w_{i,t} + \frac{1}{\mathscr{A}} \left(\xi_{i,t} - w_{i,t} \sum_{i=1}^{n} \xi_{i,t} \right), \tag{3.6}$$

where:

$$w_{i,t} = [E_t^\alpha(1-\rho_{i,t+1}^2)]^{-1} / \sum_{i=1}^{n} [E_t^\alpha(1-\rho_{i,t+1}^2)]^{-1}, \tag{3.7}$$

$$\xi_{i,t} = \frac{E_t^\alpha(\rho_{i,t+1}) u_{i,t}}{E_t^\alpha(1-\rho_{i,t+1}^2)}, \tag{3.8}$$

and $E_t^\alpha(\cdot)$ denotes the conditional expectation under a modified probability distribution such that $E_t^\alpha(\rho_{i,t+1}) = E\left[\rho_{i,t+1} \exp\left(-\mathscr{A} H_{t+1}(\alpha_t^{*\prime} y_{t+1})\right) | \underline{F_t}, \underline{y_t}\right] / E\left[\exp\left(-\mathscr{A} H_{t+1}(\alpha_t^{*\prime} y_{t+1})\right) | \underline{F_t}, \underline{y_t}\right]$. The optimal positional allocation (3.6) extends the allocation derived in (2.11) to the case of stochastic positional persistence. This allocation is a linear combination of two portfolios. The first one has positive weights $w_{i,t}$, that vary across assets as an increasing function of the modified conditional expectation $E_t^\alpha(\rho_{i,t+1}^2)$ of the squared positional persistence. If the positional persistence were time invariant, i.e. $\rho_{i,t+1} = \rho_i$, this portfolio would be the portfolio with the least risky future rank, conditional on the current values of the ranks. The second portfolio is an arbitrage portfolio (zero-cost portfolio), with weights involving the modified conditional expected ranks $E_t^\alpha(\rho_{i,t+1}) u_{i,t}$. Equation (3.6) defines the optimal positional allocation α_t^* in an implicit way, since the RHS of this equation depends on vector α_t^* through the modified conditional expectation of the positional persistence and its square.

Since the positional persistence values for most assets and dates are rather small (see Figure 8), in order to get more intuition on equation (3.6), we can consider its first-order expansion w.r.t. $\rho_{i,t+1}$. In this approximation we have $w_{i,t} \simeq 1/n$ and $\xi_{i,t} \simeq E_t(\rho_{i,t+1}) u_{i,t}$ [see Appendix 4 *vi)*], where $E_t(\rho_{i,t+1}) = E(\rho_{i,t+1} | F_t, \delta_i)$. This yields an explicit formula for the approximate optimal positional allocation:

$$\alpha^*_{i,t} = \frac{1}{n} + \frac{1}{\mathscr{A}} \left(E_t(\rho_{i,t+1}) u_{i,t} - \frac{1}{n} \sum_{i=1}^{n} E_t(\rho_{i,t+1}) u_{i,t} \right). \tag{3.9}$$

Thus, the optimal positional allocation is the linear combination of the equally weighted portfolio and an arbitrage portfolio, whose weights are given by the conditional expected ranks of the assets in deviation from their cross-sectional average. Equation (3.9) is the generalization of (2.11), for small stock-specific and time varying positional persistence.

4 Momentum strategies based on ranks

In this section we compare simple (suboptimal) positional allocation strategies which are based on the ranks of the assets and their dynamics, only.

4.1 Investment universe versus positioning universe

When analyzing a positional strategy, it is important to precisely define the investment universe, that is the set of assets potentially introduced in the portfolio, and the positional universe, that is the set of assets and portfolios used to define the rankings. For instance, for a fund (resp. fund of funds) manager, the investment universe may be a fraction of the stocks (resp. funds), whereas the positioning universe can be the set of all stocks (resp. all funds, or all funds including the funds of funds). The dynamic model for positions developed in Section 3 is appropriate for an investment universe nested in the positioning universe. In this section, we consider simple positional strategies for which both the positioning universe and the investment universe are the set of 939 stocks in our balanced panel from CRSP.

4.2 Positional momentum strategies

As a first illustration of positional strategies, let us consider momentum and contrarian (reversal) approaches [see e.g. Lehmann (1990), Jegadeesh and Titman (1993), Chan, Jegadeesh, and Lakonishok (1996)]. These strategies will be applied on the complete universe of stocks. We consider below the nine following strategies:

i) The (positional) momentum strategies denoted by PMS1 (resp. PMS2), which select an equally weighted portfolio including all stocks whose current return is in the upper 5% quantile of the CS distribution (resp., between the upper 10% and 5% quantiles), i.e. the past winners. The current (ex-post) Gaussian ranks of these stocks are such that $\hat{u}_{i,t} \geq 1.64$ (resp., $1.64 \geq \hat{u}_{i,t} \geq 1.28$). These strategies are similar to standard momentum strategies, but are based on the rank of the return on the current month, instead of the rank of the return over a longer period in the past. It is commonly believed that many stocks feature reversal in returns at a short monthly horizon [see e.g. Jegadeesh (1990) and Avramov, Chordia, and Goyal (2006)], likely due to overreaction of some investors to news[8] [De Bondt and Thaler (1985)]. Therefore, we also consider (positional) reversal strategies PRS1 (resp. PRS2), which select an equally weighted portfolio including all stocks with current rank in the lowest 5% quantile (resp., between the lower 10% and 5% quantiles), i.e. the past losers.

ii) The expected positional momentum strategies EPMS1 IN and EPMS1 OUT (resp. EPMS2 IN and EPMS2 OUT) based on the information on the rank histories. These strategies select equally weighted portfolios including the stocks with the 5% largest *expected* future ranks at each month (resp., the stocks with expected future ranks between the 5% and 10% upper quantiles). The estimated model in Section 3 is used to compute the conditional expectation of the future ranks given the current

[8] More precisely, Chan, Jegadeesh, and Lakonishok (1996) report reversal in stock returns at short horizons smaller than 6 months, and for periods between 3 and 5 years, but momentum in returns between 6 months and 3 years.

information. As the positional factor ($F_{p,t}$) is assumed i.i.d. over time, the expected future rank of asset i is given by $E_t(u_{i,t+1}) = \bar{\rho}_i u_{i,t}$, where the expected positional persistence $\bar{\rho}_i = E[\Psi(\beta_i + \gamma_i F_{p,t+1})|\beta_i, \gamma_i]$ involves the expectation with respect to the historical distribution of $F_{p,t+1}$. In our numerical implementation, the expectation is replaced by a sample average over the factor estimates $\hat{F}_{p,t+1}$, the stock-specific effects are replaced by the estimates $\hat{\beta}_i$ and $\hat{\gamma}_i$, and the ex-ante current rank is replaced by the ex-post rank $\hat{u}_{i,t}$. For the in-sample strategies EPMS1 IN and EPMS2 IN the entire available sample of returns from January 1990 to December 2010 is used to estimate the factor model. On the other hand, in order to assess the out-of-sample performance, for the out-of-sample strategies EPMS1 OUT and EPMS2 OUT the model is re-estimated at each month using a rolling window of 10 years of data. The expected future ranks determining the EPMS allocation are computed using these rolling estimates.

iii) As a benchmark, we also consider the market portfolio defined as the equally weighted portfolio computed on all stocks.

We provide in Figure 9 the ex-post properties of these portfolios over the period from January 2000 to December 2009.

[FIGURE 9 : Ex-post properties of the portfolio strategies, 2000-2009.]

Panel (a) provides the evolution of the Gaussian ranks for the management strategies PMS2, PRS1, EPMS1 OUT, EPMS2 OUT, and the equally weighted portfolio, and panels (b) and (c) the evolution of their excess returns and cumulated returns over the period. The series of Gaussian ranks of the equally weighted portfolio is less disperse than the others. For ease of comparison, we provide historical summary statistics of Gaussian ranks and returns in Table 1.

[TABLE 1 : Ex-post properties of the portfolio strategies, 2000-2009.]

Even if the standard financial theory suggests that the market portfolio has some efficiency properties, we observe that it is not systematically well ranked, or with the highest return. The historical average of the Gaussian ranks of the equally weighted portfolio and momentum strategies PMS1 and PMS2 are slightly larger than 0. Thus, on average, the return of these strategies is slightly above the CS median, while the historical averages of the ranks of the reversal and expected positional momentum strategies are larger. The largest average Gaussian rank is featured by the positional strategy EPMS1 IN, and is equal to 0.22, followed by PRS1, EPMS2 IN and EPMS2 OUT with 0.20, 0.17 and 0.14, respectively. As expected from the discussion in Section 3.3, the equally weighted portfolio is close to the median rank. The Sharpe ratios of the expected positional momentum strategies in sample are the largest ones (both higher than 1.10), while PRS1 has a Sharpe ratio (0.96) only slightly larger than EMPS2 OUT (0.94). All positional strategies based on the 5%-10% quantile range are less volatile than the corresponding positional strategies based on the first 5% quantile since they avoid extreme effects. However, while this fact results in a larger Sharpe ratio for EPMS2 OUT compared to EPMS1 OUT, for strategies based on expected future ranks in-sample and reversal strategies, considering the upper 5%-10% quantile range yields a smaller Sharpe ratio than considering the upper 5% quantile. The ranks of the reversal strategy PRS1 are the most volatile ones. Moreover, the series of excess returns of the reversal strategies are negatively skewed and feature the largest negative values, as can be deduced by the historical 5% quantile. To summarize, the out-of-sample expected positional momentum strategy based on the 5%-10% quantile range performs almost as good as the reversal strategy PRS1 in terms of Sharpe ratio, but has substantially smaller downside risk. This is made possible by the combination of momentum and reversal type of allocations across stocks, that is implicit in this strategy. Moreover, the in- and out-of-sample expected positional momentum strategies outperform the equally weighted portfolio, both in terms of average Gaussian rank and Sharpe ratio.

Finally, panel (d) in Figure 9 and the bottom part of Table 1 give some insight on the asset turnover of the portfolio strategies. The asset turnover is measured by the unweighted proportion of selected stocks which are not kept in the portfolio between two consecutive dates. This unweighted measure of asset turnover provides an information on the potential transaction costs of the portfolio updating. However, this information is rather crude, since it does not account for the variation in the quantities of assets included in the portfolio. For instance, there is no unweighted asset turnover in the equally weighted portfolio, but the transaction costs are not zero for this portfolio, as rebalancing is required to keep the relative allocations in value constant at $1/n$. Among the eight reversal and momentum strategies, the ones based on expected future ranks in the first 5% quantile, both in-sample and out-of-sample, have the smallest average asset turnover.

5 The full-fledged model

The positional portfolio strategies implemented in Section 4 rely on the dynamics of the individual ranks only. However, the optimal positional allocation defined in Section 2 generally involves the entire distribution of the return histories. In fact, the portfolio rank is a nonlinear aggregate of the individual ranks depending on the cross-sectional return distribution. Two features have to be considered in order to pass from the dynamics of the ranks to the dynamics of the returns (see Scheme 1 in the Introduction). First, we have to specify the cross-sectional distribution in a flexible way. This cross-sectional distribution varies in time as a function of macro factors. Second, we have to explain how these macro factors, that impact the cross-sectional distribution, are linked to the positional factor, that drives the persistence of ranks dynamics. To get a tractable model, we assume in Section 5.1 that the cross-sectional distributions belong to the Variance-Gamma (VG) family (see Appendix 5 for a review on the VG family). The macro factors are time varying parameters characterizing the distributions in this family. Next, in Section 5.2 we specify the joint dynamics of the positional and distributional macro-factors by a Gaussian Vector Autoregressive (VAR) model.

5.1 Specification of the cross-sectional distributions

Let us first complete the analysis of Section 2 by investigating if the empirical CS distributions are close to Gaussian distributions, and studying how empirical CS summary statistics, such as mean, standard deviation, skewness and kurtosis, vary over time. In Figures 10 and 11, we provide the empirical CS distributions and their Gaussian approximations at some months. In particular, in Figure 11 we focus on the period around the 2008 Lehman Brothers bankruptcy.

[FIGURE 10: Cross-sectional distributions of monthly CRSP stock returns.]

[FIGURE 11: Cross-sectional distributions around the 2008 Lehman Brothers bankruptcy.]

The comparison between the panels in Figure 10 shows that the empirical CS distribution may be close to a Gaussian in some months (e.g. August 1998), may feature rather fat tails (e.g. July 1995, December 2006), or be asymmetric (e.g. November 2000). In Figure 11, we see that in July and August 2008, before the Lehman Brothers crisis, the CS distribution is non-normal, with a peak close to 0 and is slightly right-skewed. Instead, in October 2008, the month after Lehman Brothers filed for Chapter 11 bankruptcy protection (September 15, 2008), the CS distribution is close to Gaussian with a large negative mean of about -18%.

The above empirical evidence shows that it is necessary to choose the cross-sectional distributions in an extended family including the Gaussian family as a special case, and to introduce additional macro-factors accounting for time-varying higher-order moments. We consider in our analysis the

Variance-Gamma (VG) family. The distributions in this family are indexed by four parameters, that are in a one-to-one relationship with the mean μ_t, the log-volatility $\log \sigma_t$, the skewness s_t and the log excess kurtosis $\log k_t^*$, where $k_t^* = k_t - 3(1 + s_t^2/2)$ and k_t denotes the kurtosis (see Appendix 5). The excess kurtosis k_t^* is a measure of the fatness of the tails of the CS distribution of returns at month t, in excess of $3(1 + s_t^2/2)$. The latter value is the minimum admissible kurtosis for a VG distribution with skewness parameter s_t. Since the above four transformed parameters can vary independently on the entire real line, they are chosen to define the vector of distributional macro-factors:

$$F_{d,t} = (\mu_t, \log \sigma_t, s_t, \log k_t^*)'. \tag{5.1}$$

We provide in Figure 12 the time series of estimated distributional macro-factor values $\hat{F}_{d,t}$, obtained from the empirical CS moments, along with their asymptotic (large n) pointwise 95 % confidence bands [9].

[FIGURE 12: Time series of estimated distributional macro-factors.]

The series of the cross-sectional mean (Panel (a)) is rather close to the return series of the CRSP Equally Weighted Index (not shown). The log CS standard deviation (Panel (b)) is larger around crisis periods, namely in 1991 (Gulf crisis), 1998 (LTCM crisis), 2000-2001 (tech bubble) and 2008-2009 (the subprime crisis). A value of factor $\log \sigma_t$ close to -2 corresponds to a standard deviation of the CS distribution of returns of about 13.5%. The CS skewness is mostly positive, that is, the CS distributions are often right skewed. The log excess kurtosis varies between 1 and -6, which correspond to excess kurtosis values close to 3 and 0 respectively. The series of CS skewness and kurtosis can be used to compute the Jarque-Bera statistic for the CS distribution of returns for each month. Crisis periods are among the months characterized by the smallest values of the CS Jarque-Bera statistic, that are months

[9] The asymptotic standard errors of the distributional macro-factors are computed with the results in Bai and Ng (2005).

in which the CS distribution is closer to a Gaussian one [see panel (b) of Figure 10 for August 1998 (LTCM Crisis), and Panel (d) of Figure 11 for October 2008 (Lehman Brothers crisis)]. This feature has already been noted by the econophysics literature for daily returns [see e.g. Borland (2012)].

5.2 The factor dynamics

Let us now specify the factor dynamics. We assume that the joint vector of distributional and positional macro-factors $F_t = (F'_{d,t}, F_{p,t})'$ follows a 5-dimensional Gaussian Vector Autoregressive process of order 1 [VAR(1)]:

$$F_t = a + AF_{t-1} + \eta_t, \qquad \eta_t \sim IIN(0, \Sigma), \tag{5.2}$$

where a is the vector of intercepts, A is the matrix of autoregressive coefficients, and Σ is the variance-covariance matrix of the innovations. We estimate parameters a, A and Σ in the joint VAR dynamics in equation (5.2) after replacing the unobservable values of the positional and distributional macro-factors with their estimates $\hat{F}_t = (\hat{F}'_{d,t}, \hat{F}_{p,t})'$, where $\hat{F}_{d,t}$ is defined in Section 5.1 and $\hat{F}_{p,t}$ is defined in equations (3.4)-(3.5).

In Table 2 we present the parameter estimates for the macro-factor VAR dynamics with their standard errors in parentheses. We also provide the estimated correlation matrix of the innovations vector.

[TABLE 2 : Estimates of the VAR (1) model for the macro-factor process.]

Five coefficients in the estimated autoregressive matrix are statistically significant (at the 1% level). As expected, the autoregressive coefficient of the log CS standard deviation is significant and large (0.84) pointing to a strong serial persistence in the dispersion of the CS distribution. The CS mean also features positive serial persistence, with estimated autoregressive coefficient 0.32. This multivariate regression coefficient has to be compared with the univariate autoregressive coefficient of the monthly return series of the CRSP Equally Weighted Index, that is equal to 0.28 in our sample period. We find a strong evidence for the analog of the Black leverage effect [Black (1976)], namely a negative regression coefficient of the current log CS standard deviation on the past CS mean return equal to -0.79. The estimated coefficient -0.61 of log CS excess kurtosis on lagged log CS standard deviation suggests that the tails in the CS distribution get thinner after a month characterized by a positive shock on the CS dispersion. This effect is likely related to the finding that the CS distribution is close to Gaussian in crisis periods. The multivariate autoregressive coefficient for the log CS excess kurtosis in Table 2 is not statistically significant. However, from the clustering in fat tails of the CS distributions observed in Figures 1 and 2, the CS (excess) kurtosis features serial persistence. Indeed, the univariate autoregressive coefficient of log CS excess kurtosis is equal to 0.37 and is statistically significant. The difference between the univariate and multivariate autoregressive coefficients is explained by the dynamic link between log excess CS kurtosis and log CS standard deviation, and by the contemporaneous correlation between the innovations on these two series. The autoregressive coefficient of the positional factor, and its regression coefficients on the lagged values of the cross-sectional factor, are not statistically significant. This finding is compatible with the marginal white noise property of the positional factor found empirically in Section 3.2 and used in Section 4. The eigenvalues of the estimated autoregressive matrix are 0.811, and two pairs of complex conjugate eigenvalues with modulus 0.187 and 0.071, respectively. Thus, the modulus of all eigenvalues is smaller than 1, which implies the stationarity of the estimated VAR process of the macro-factors driving both the dynamics of ranks and the dynamics of the cross-sectional distribution.

All the estimated contemporaneous covariances of the shocks are significantly different from 0. In particular, we observe a negative contemporaneous correlation equal to -0.32 between the shocks on CS mean and the positional persistence factor. Thus, a small cross-sectional mean of returns tends to be associated with a large positional persistence for those stocks having positive loadings on the positional factor. Such stocks are the majority in our sample (see Figure 6). The factor $F_{d,t}$ driving the univariate CS distributions and the factor $F_{p,t}$ driving the positional persistence are not independent. [10]

6 Efficient positional strategies

Let us now implement the efficient positional strategies defined in Section 2.2 using the complete dynamic model. Since the efficient positional or mean-variance management strategies demand the inversion of a n-by-n matrix, they can only be applied with a limited number n of assets, significantly smaller than 939. We apply these strategies to an investment universe corresponding to the $n = 57$ stocks in the industrial sector of utilities. The strategies are the following ones: *i)* The efficient positional strategy (EPS), with CARA positional utility function and positional risk aversion $\mathscr{A} = 3$. We implement the EPS strategy both in-sample (EPS IN) by using the entire available sample of returns from January 1990 to December 2010 to estimate the factor model, and out-of-sample (EPS OUT) by using an expanding window of past returns for estimation from January 1990 to the investment date; *ii)* The positional momentum strategy (PMS) based on the 20 stocks with the largest current ranks; *iii)* The positional reversal strategy (PRS) based on the 20 stocks with the smallest current ranks; *iv)* An expected positional momentum strategy (EPMS) based on the 20 stocks with the largest expected future ranks; *v)* The sectoral equally weighted (EW) portfolio; *vi)* The standard mean-variance (MV) strategy based on the unconditional moments. The financial literature reports poor out-of-sample properties for the MV strategy, which is often outperformed by the minimum-

[10] The point estimate of the conditional variance of factor $F_{p,t}$ is slightly larger than 1. By taking into account the standard deviation of the estimate, this is compatible with the normalization of unconditional variance equal to 1.

variance portfolio strategy [see e.g. Jagannathan and Ma (2003)]. For this reason, we include also the unconditional minimum-variance (MinV) strategy in our comparison. We implement strategies EMPS, MV and MinV out-of-sample [11].

We give in Appendix 6 a numerical algorithm to solve the constrained maximization problem (2.4) defining the EPS strategy. The algorithm involves the inversion of a $n \times n$ Hessian matrix, and computational costs grow quadratically in the number of stocks n in the investment universe. This explains the choice to restrict the investment universe compared to Section 4. An alternative optimization algorithm with linearly growing computational costs could be obtained by replacing the above Hessian matrix with a diagonal matrix having the same diagonal elements.

We provide in Figure 13 the time series of cumulated portfolio excess returns for the above strategies. Summary statistics of the Gaussian ranks and excess returns series are presented in Table 3.

[TABLE 3 : Ex-post properties of the portfolio strategies, utilities sector, 2000-2009.]

[FIGURE 13 : Time series of cumulative returns of the portfolio strategies, utilities sector, 2000-2009.]

We consider different criteria to compare the performances of the strategies. They include: *i)* the mean and standard deviation of the Gaussian ranks, and the average positional utility; *ii)* the frequency of returns above a certain cross-sectional quantile of the investment universe; *iii)* summary statistics and Sharpe ratios of the excess returns. The in-sample EPS strategy outperforms the other allocation strategies according to most criteria. This finding is also confirmed by the series of the cumulated returns in Figure 13. The EPS strategy implemented out-of-sample outperforms the PMS, PRS, MV

[11] Given the relatively large cross-sectional dimension, we use a shrinkage estimator for the variance-covariance matrix of excess returns in both the mean-variance and minimum-variance strategies, as proposed by Ledoit and Wolf (2003). This estimator consists in an optimally weighted average of the sample covariance matrix of the excess returns and a single-index covariance matrix.

and MinV strategies according to both the average positional utility and Sharpe ratio. Along those dimensions, EPS OUT and the equally weighted portfolio perform similarly, but the former ensures larger probabilities to be well-ranked. For instance, the returns of the EPS OUT strategy is about 60% of the times above the CS median of the returns in the investment universe, and about 10% of the times above the CS 60% quantile. Instead, the equally weighted portfolio is above the CS 60% quantile only in 3% of the months. The standard (positional) momentum and reversal strategies PMS and PRS ensure large probabilities to be well ranked, but feature Gaussian ranks that are among the most volatile ones. A similar remark can be done for the MV strategy, which provides a large probability to be in the top 30%, but a very small value of the expected positional utility. In fact the MV strategy alternates very high ranks and very low ranks and appears as very risky. These remarks explain the low average positional utility of the PMS, PRS and MV strategies. Similarly as in Section 4, the EPMS strategy outperforms the PMS along all criteria, and features the largest Sharpe ratio. The EPMS overperforms also the PRS concerning the expected positional utility and the Sharpe ratio. In fact, the reversal strategy PRS, which is found to perform rather well for the larger investment universe in Section 4, features the smallest Sharpe ratio among the positional strategies for the investment universe of the utilities. The overperformance of the EPMS compared to traditional momentum and reversal strategies is likely due to the ability of the former strategy to exploit the time-varying and stock-specific positional persistence (see Figures 6 and 7). Indeed, the traditional momentum (resp. reversal) strategies implicitly assume that all stocks feature a positive (resp. negative) positional persistence, that is constant through time. Instead, the EPMS provides a combination of momentum and reversal strategies based on the stock-specific and time-specific information.

In order to better understand the similarities in the performances of the efficient positional strategy and the EW portfolio, in Figure 14 we display the relative discrepancy between the allocation vectors implied by these strategies overtime.

[FIGURE 14: Observed measure of relative discrepancy of optimal positional allocation from EW portfolio for the subsample of utilities]

The relative discrepancy at a given month t is measured as $n\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\alpha_{i,t}^{*}-1/n)^2}$, where the $\alpha_{i,t}^{*}$ are the efficient positional allocations in (2.4). This measure corresponds to the ratio between the standard deviation $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\alpha_{i,t}^{*}-1/n)^2}$ and the mean $1/n$ of the allocations across assets in the efficient positional portfolio. The relative discrepancy varies overtime between 0.4 and 1.2. In panel (b) of Figure 14 we see that the discrepancy increases when the predicted value $E(F_{p,t+1}|F_t)$ of the positional factor $F_{p,t+1}$, based on the full vector of macro-factors F_t, becomes larger in absolute value. We can understand qualitatively the pattern in panel (b) of Figure 14 by means of equation (3.9), which provides an approximation of the efficient positional allocation when the CS distribution of returns is close to Gaussian and the positional persistence is small. From (3.9), the quantity $\frac{1}{n}\sum_{i=1}^{n}(\alpha_{i,t}^{*}-1/n)^2$ is equal to the empirical cross-sectional variance of the conditional expected ranks, divided by $\mathscr{A}^2$. When the investment universe is large, i.e. n tends to infinity, the Law of Large Numbers (LLN) implies that this empirical cross-sectional variance converges to its theoretical counterpart, and we get [see Appendix 4 *vii)*]:

$$\frac{1}{n}\sum_{i=1}^{n}(\alpha_{i,t}^{*}-1/n)^2 \simeq \frac{1}{\mathscr{A}^2}E[(\beta_i+\gamma_i E(F_{p,t+1}|F_t))^2\,|F_t], \tag{6.1}$$

where the expectation in the RHS is w.r.t. the distribution of the random effects (β_i,γ_i). We get a quadratic function of the conditional expectation $E(F_{p,t+1}|F_t)$ of the positional factor. This quadratic function is minimized when the macro-factor vector F_t is such that $E(F_{p,t+1}|F_t) = -E[\beta_i\gamma_i]/E[\gamma_i^2]$. From Figures 6 and 7, the individual effect β_i has a mean close to zero, and the covariance between the individual effects β_i and γ_i is positive, which explains why the minimum of the discrepancy measure in panel (b) of Figure 14 is attained for a negative value of the predicted positional factor.

Finally, Table 3 also provides information on the turnover of the strategies. The measure of the turnover is now weighted to account for the different weights introduced in an efficient portfolio, and is defined as follows: $Turnover_t = \sum_{i=1}^{57} |\alpha_{i,t} - \alpha_{i,t-1}|$, where $\alpha_{i,t}$ is the relative weight of stock i at date t for a certain strategy. Among the positional strategies, EPS OUT has the lowest and less volatile turnover.

7 Conclusions

In this paper we introduce different positional portfolio allocation strategies, that are the expected positional momentum strategies (EPMS) and the efficient positional strategies (EPS). We consider these allocation strategies in a big data framework, in which the investment universe consists of hundreds, or even thousands, of stocks. The implementation of expected positional momentum strategies simply requires a dynamic model for the ranks. This model is used to detect at each date the stocks with high expected future ranks, and the ones with low expected future ranks. This information is implicitly used in the EPMS strategies to mix in an efficient way momentum and reversal (or contrarian) strategies and leads to a closed-form characterization of the EPMS portfolios. The implementation of the efficient positional strategies is more demanding, since these strategies require a complete dynamic model for both the ranks and the cross-sectional distributions of returns. Moreover, the necessity of solving a constrained optimization problem implies that EPS can be applied only with more limited number of assets (57 in our example). As expected, these positional strategies have good properties in terms of the position of the portfolio returns. More surprising are their rather nice properties concerning the portfolio returns themselves. The main reason is that these strategies based on positions are robust to abnormal returns. It is well known that the standard mean-variance allocation strategy is very sensitive to outliers, especially when it is applied with a large number of assets. In particular, its performance can be much worse than the performance of the naive equally weighted portfolio, or $1/n$-strategy, giving the impression that sophisticated allocation strategies are not useful. Our analysis shows that, indeed, the equally weighted portfolio is difficult to outperform for portfolios invested in stocks. For instance, the $1/n$-strategy is clearly competitive with other strategies as the basic momentum, reversal and min-variance strategies. However, the positional strategies outperform the equally-weighted portfolio for performance criteria based on positions.

The positional strategies considered in this paper can be extended in various ways. For instance, we can consider a fund manager interested jointly in different rankings. Then, he or she will optimize a positional utility function depending on these different ranks associated with different universes. It is possible to manage jointly the ranking among the funds of the same management style and the ranking among all funds, and to weight differently the two associated universes. In the perspective of an analysis applied to funds managers' behaviors, it would be interesting to develop inference methods to test if the managers follow positional strategies, and to estimate their selected positioning universes and positional risk aversion parameters. Moreover, as noted in the mutual fund literature, if the rankings are published at the end of each year, the fund managers compete in annual tournaments that begin in January and end in December. They could follow a standard management at the beginning of the year and pass to a positional management with more risk in the second part of the year, if they performed poorly in the first part of the year [see Brown, Harlow, and Starks (1996), Chen and Pennacchi (2009) and Schwarz (2012)]. These questions are left for future research. Likely, the analysis will encounter an identification problem, especially if several fund managers follow such endogenous strategies [see the reflection problem highlighted by Manski (1993)].

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TABLES

Table 1: Ex-post properties of the portfolio strategies, 2000-2009.

	PMS1	PMS2	PRS1	PRS2	EPMS1 IN	EPMS2 IN	EPMS1 OUT	EPMS2 OUT	EW
Gaussian ranks									
Mean	0.0276	0.0258	0.2043	0.1318	0.2171	0.1703	0.1105	0.1360	0.0646
St. dev.	0.4591	0.3574	0.4783	0.3748	0.4762	0.3181	0.4179	0.2587	0.1059
Excess returns									
Mean	0.0061	0.0059	0.0228	0.0158	0.0230	0.0184	0.0138	0.0155	0.0088
St. dev.	0.0690	0.0566	0.0818	0.0709	0.0681	0.0565	0.0680	0.0573	0.0490
Sharpe ratio (ann.)	0.3041	0.3587	0.9667	0.7736	1.1706	1.1295	0.7017	0.9357	0.6244
Skew.	0.1896	0.4073	-0.0891	-0.5103	0.2988	0.0581	0.1986	-0.1735	-0.6151
Exc. kurt.	1.1989	1.9350	1.1367	1.7235	1.1672	1.8834	1.5131	1.6940	2.2703
Quant. 5%	-0.1208	-0.0751	-0.1405	-0.1161	-0.0783	-0.0762	-0.1042	-0.0809	-0.0898
Quant. 25%	-0.0343	-0.0334	-0.0185	-0.0144	-0.0152	-0.0136	-0.0284	-0.0176	-0.0171
Quant. 50%	0.0106	0.0054	0.0214	0.0159	0.0203	0.0196	0.0141	0.0195	0.0093
Quant. 75%	0.0492	0.0488	0.0717	0.0573	0.0587	0.0508	0.0572	0.0498	0.0418
Quant. 95%	0.1010	0.0801	0.1445	0.1216	0.1372	0.1042	0.1119	0.1030	0.0737
Turnover									
Turn mean	0.8971	0.9415	0.8931	0.9356	0.8214	0.9080	0.8248	0.9094	0.0000
Turn std	0.0962	0.0933	0.1070	0.0927	0.1073	0.0931	0.0988	0.0940	0.0000

The table provides summary statistics for the monthly series of the Gaussian ranks, and of the excess returns, for the eight portfolio allocation strategies PMS1, PMS2, PRS1, PRS2, EPMS1 IN, EPMS2 IN, EPMS1 OUT and EPMS2 OUT, and for the equally weighted portfolio (EW), in the period 2000/1 - 2009/12. Strategy PMS1 (resp. PMS2) selects an equally weighted portfolio of all stocks whose current return is in the upper 5% quantile of the CS distribution (resp. between the upper 10% and 5% quantiles). Strategy PRS1 (resp. PRS2) selects an equally weighted portfolio of all stocks whose current return is in the lower 5% quantile of the CS distribution (resp. between the lower 10% and 5% quantiles). Strategy EMPS1 IN (resp. EPMS2 IN) selects an equally weighted portfolio of all stocks with the 5% largest expected future rank (resp., with the expected future rank between the upper 5% and 10% quantiles), with the parameters of the model estimated on the full sample (1990/1-2009/12). Strategy EMPS1 OUT (resp. EPMS2 OUT) selects an equally weighted portfolio of all stocks with the 5% largest expected future rank (resp., with the expected future rank between the upper 5% and 10% quantiles), with the parameters of the model estimated on the available sample up to the investment date. The investment universe consists of all the $n = 939$ NYSE, AMEX and NASDAQ stocks in our sample (see Section 2 for a description). The ranks are computed w.r.t. the CS distribution of the monthly returns of all the stocks in our sample. The Sharpe ratio is annualized. The table also provides the mean and the standard deviation of turnover. The turnover is measured by the proportion of selected stocks which are not kept in the portfolio between two consecutive dates.

Table 2: Estimates of the VAR (1) model for the macro-factor process.

The model for the dynamics of the factor $F_t = (\,\mu_t, \log\sigma_t, s_t, \log k_t^*, F_{p,t}\,)'$ is the Gaussian VAR(1) process:

$$F_t \;=\; a + AF_{t-1} + \eta_t, \qquad \eta_t \sim IIN(0,\Sigma)\,.$$

The estimates for the period 1990/01 - 2010/12 are given by:

$$\hat{a} = \begin{bmatrix} 0.0005 \\ (\mathit{0.0342}) \\ -0.3834^{***} \\ (\mathit{0.0948}) \\ 0.1084 \\ (\mathit{0.3787}) \\ -0.8249^{***} \\ (\mathit{0.2729}) \\ -0.1230 \\ (\mathit{0.8227}) \end{bmatrix}, \qquad \hat{A} = \begin{bmatrix} 0.3156^{***} & -0.0076 & -0.0096 & -0.0117 & -0.0026 \\ (\mathit{0.0914}) & (\mathit{0.0155}) & (\mathit{0.0080}) & (\mathit{0.0091}) & (\mathit{0.0029}) \\ -0.7889^{***} & 0.8406^{***} & -0.0128 & 0.0369 & 0.0066 \\ (\mathit{0.2532}) & (\mathit{0.0430}) & (\mathit{0.0221}) & (\mathit{0.0251}) & (\mathit{0.0080}) \\ 2.9916^{***} & -0.0966 & 0.0023 & -0.0911 & -0.0252 \\ (\mathit{1.0113}) & (\mathit{0.1718}) & (\mathit{0.0882}) & (\mathit{0.1004}) & (\mathit{0.0321}) \\ 0.2588 & -0.6060^{***} & -0.0427 & -0.0211 & -0.0294 \\ (\mathit{0.7290}) & (\mathit{0.1238}) & (\mathit{0.0636}) & (\mathit{0.0723}) & (\mathit{0.0232}) \\ -0.3748 & -0.0341 & -0.2034 & 0.1702 & -0.0112 \\ (\mathit{2.1973}) & (\mathit{0.3732}) & (\mathit{0.1917}) & (\mathit{0.2181}) & (\mathit{0.0698}) \end{bmatrix},$$

$$\hat{\Sigma} = \begin{bmatrix} 0.0017^{***} & & & & \\ (\mathit{0.0001}) & & & & \\ 0.0009^{***} & 0.0134^{***} & & & \\ (\mathit{0.0002}) & (\mathit{0.0009}) & & & \\ 0.0128^{***} & 0.0125^{***} & 0.2134^{***} & & \\ (\mathit{0.0011}) & (\mathit{0.0025}) & (\mathit{0.0138}) & & \\ -0.0015^{**} & -0.0153^{***} & -0.0359^{***} & 0.1109^{***} & \\ (\mathit{0.0006}) & (\mathit{0.0019}) & (\mathit{0.0072}) & (\mathit{0.0072}) & \\ -0.0135^{***} & 0.0177^{***} & -0.0574^{***} & -0.0416^{***} & 1.0073^{***} \\ (\mathit{0.0020}) & (\mathit{0.0054}) & (\mathit{0.0213}) & (\mathit{0.00154}) & (\mathit{0.0650}) \end{bmatrix},$$

with standard errors in parentheses. The asterisks ***, **, * denote that the coefficient is significant at the 1%, 5%, and 10% levels, respectively. The correlation matrix corresponding to the estimated variance-covariance matrix $\hat{\Sigma}$ is:

$$\begin{bmatrix} 1.0000 & & & & \\ 0.1892 & 1.0000 & & & \\ 0.6614 & 0.2348 & 1.0000 & & \\ -0.1075 & -0.3979 & -0.2336 & 1.0000 & \\ -0.3211 & 0.1528 & -0.1238 & -0.1246 & 1.0000 \end{bmatrix}.$$

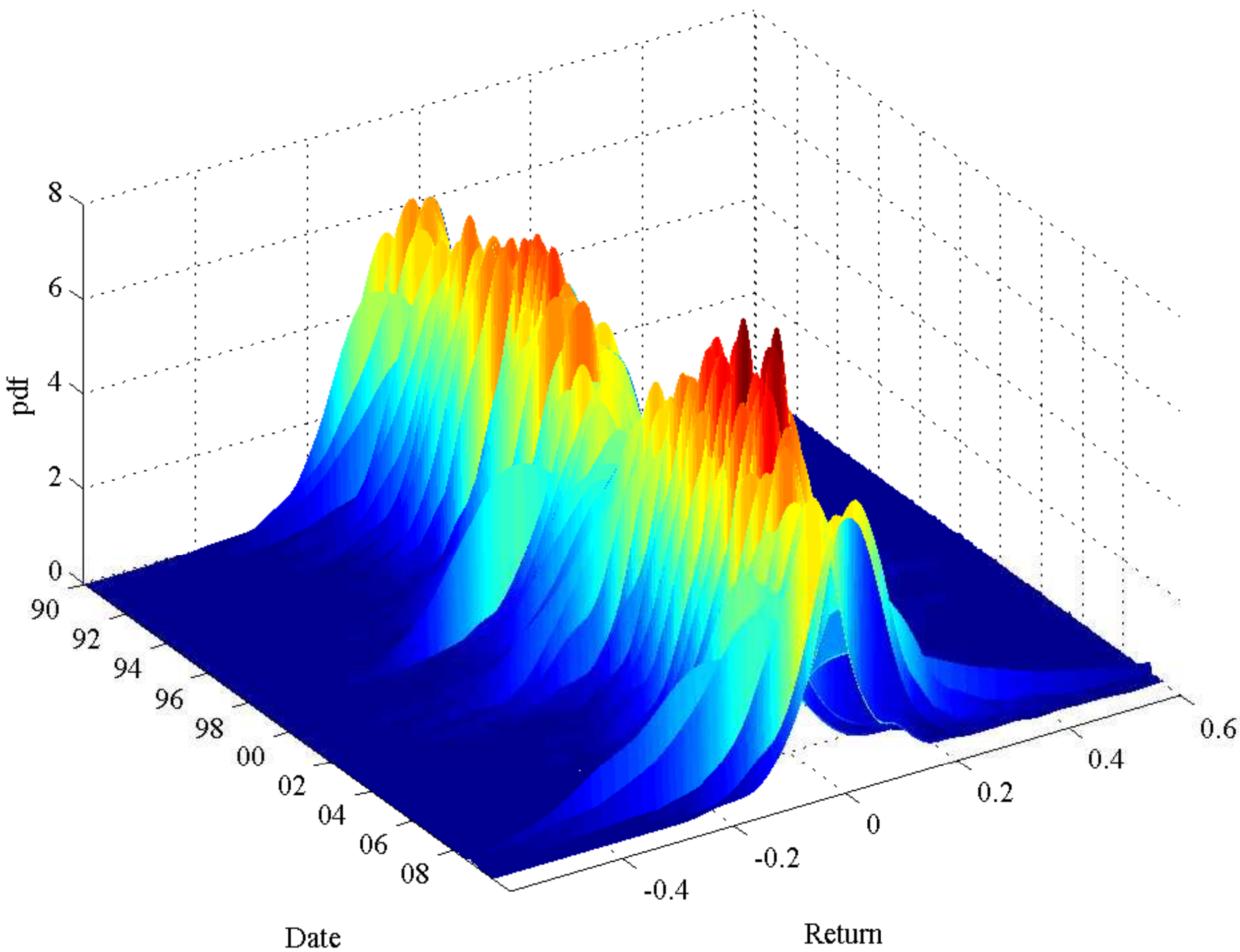
Table 3: Ex-post properties of the portfolio strategies, utilities sector, 2000-2009.

	EPS IN	EPS OUT	PMS	PRS	EPMS	EW	MV	MinV
Gaussian ranks								
Mean	0.0906	0.0534	0.0484	0.0589	0.0678	0.0566	-0.0556	0.0119
St. dev.	0.4090	0.4034	0.4406	0.4437	0.4224	0.4043	0.7532	0.4252
E(pos. utility)	-1.6012	-1.7596	-2.0378	-1.9668	-1.9388	-1.7471	-22.6449	-2.2134
Top positions								
$>$ 50% quant.	0.7000	0.6083	0.4833	0.6083	0.5833	0.6167	0.4000	0.4667
$>$ 60% quant.	0.1750	0.0917	0.1917	0.2833	0.2667	0.0333	0.3000	0.2750
$>$ 70% quant.	0.0250	0.0083	0.0667	0.0667	0.0500	0.0000	0.2500	0.2083
Excess returns								
Mean	0.0104	0.0076	0.0076	0.0076	0.0083	0.0079	0.0006	0.0052
St. dev.	0.0424	0.0410	0.0414	0.0476	0.0433	0.0414	0.0611	0.0339
Sharpe Ratio (ann.)	0.8529	0.6436	0.6371	0.5532	0.6614	0.6593	0.0353	0.5344
Skew.	-0.5238	-0.7922	-0.4227	-0.8576	-0.8488	-0.7210	0.1965	-0.2710
Exc. kurt.	1.9679	1.3576	1.3186	1.1699	1.6654	1.1882	2.2707	3.4109
Quant. 5%	-0.0759	-0.0720	-0.0735	-0.0818	-0.0754	-0.0703	-0.1074	-0.0503
Quant. 25%	-0.0090	-0.0111	-0.0135	-0.0099	-0.0100	-0.0138	-0.0320	-0.0137
Quant. 50%	0.0142	0.0124	0.0134	0.0142	0.0134	0.0141	-0.0013	0.0066
Quant. 75%	0.0381	0.0327	0.0321	0.0355	0.0358	0.0326	0.0336	0.0219
Quant. 95%	0.0701	0.0702	0.0706	0.0719	0.0683	0.0692	0.1028	0.0527
Turnover								
Mean	0.5745	0.4825	1.2379	1.1833	1.1818	0.0000	0.4398	0.0862
St. dev.	0.1559	0.1192	0.2378	0.2230	0.2953	0.0000	0.2636	0.1109

The table provides summary statistics for the monthly series of the Gaussian ranks, and of the excess returns, for the eight portfolio allocation strategies with investment universe being 57 stocks in the utilities sector in the period 2000/1 - 2009/12. The efficient positional strategy EPS IN uses the model estimated on the full sample (1990/1-2009/12). EPS OUT is the efficient positional strategy based on the model estimated on the available sample up to the investment date. The positional utility is a CARA function with positional risk aversion parameter $\mathscr{A} = 3$. The strategy PMS (resp. PRS) selects an equally weighted portfolio of the 20 stocks with largest current ranks (resp., smallest current ranks). The strategy EPMS selects an equally weighted portfolio of the 20 stocks with largest expected future ranks. The ranks are computed w.r.t. the CS distribution of the monthly returns of all the NYSE, AMEX and NASDAQ stocks in our sample. Strategies MV and MinV are mean-variance and minimum-variance strategies implemented using a shrinkage estimator for the variance-covariance matrix of excess returns. E(pos. utility) is the time series average of the positional utility of the portfolio returns. In the panel denoted "Top positions" we report the observed frequency of portfolio returns above a certain cross-sectional quantile of the stock returns in the investment universe. The Sharpe ratio is annualized. The table also provides the mean and the standard deviation of turnover, computed at each month t as follows: Turnover$_t = \sum_{i=1}^{n} |\alpha_{i,t} - \alpha_{i,t-1}|$, where $\alpha_{i,t}$ is the weight of stock i at date t.

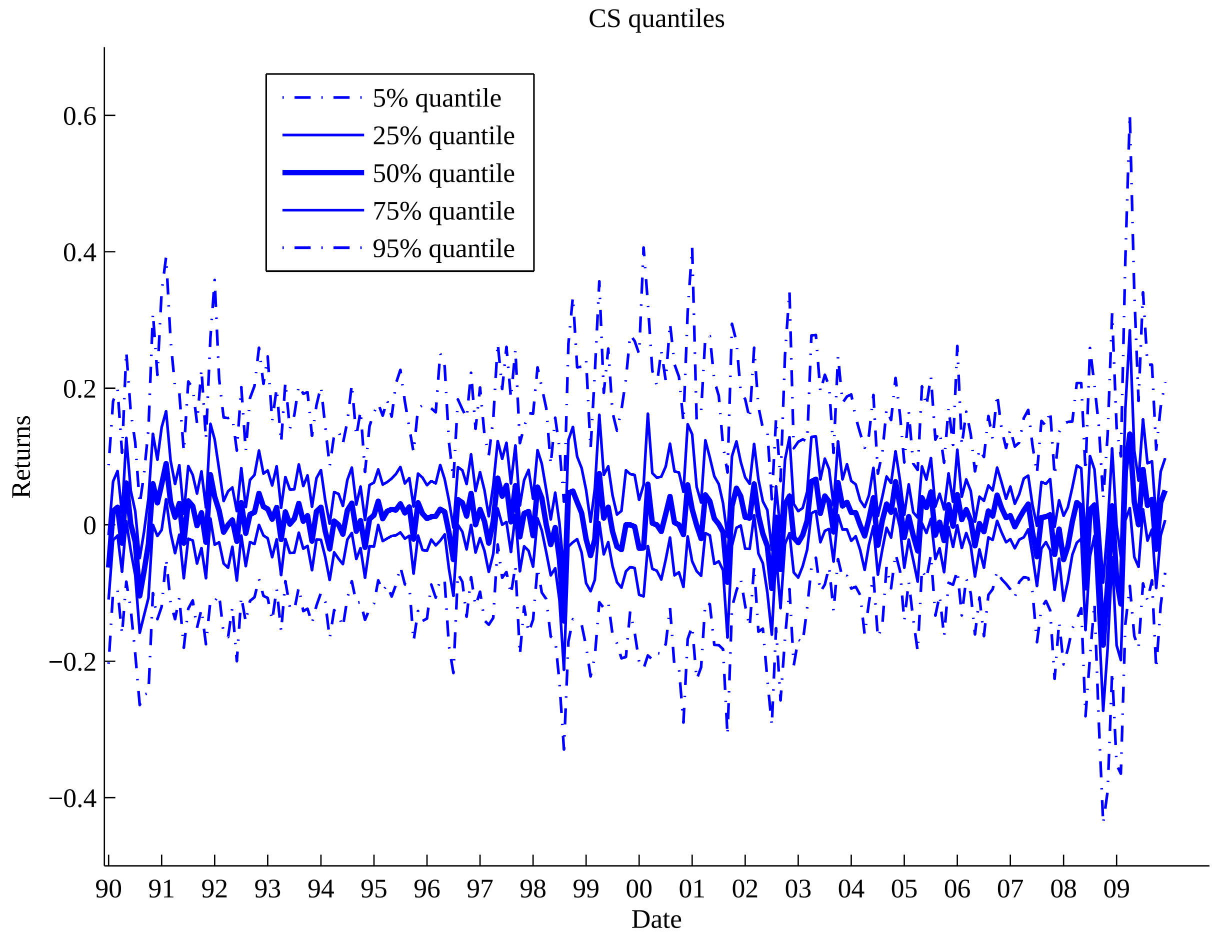
FIGURES

Figure 1: Time series of cross-sectional distributions of monthly CRSP stock returns.



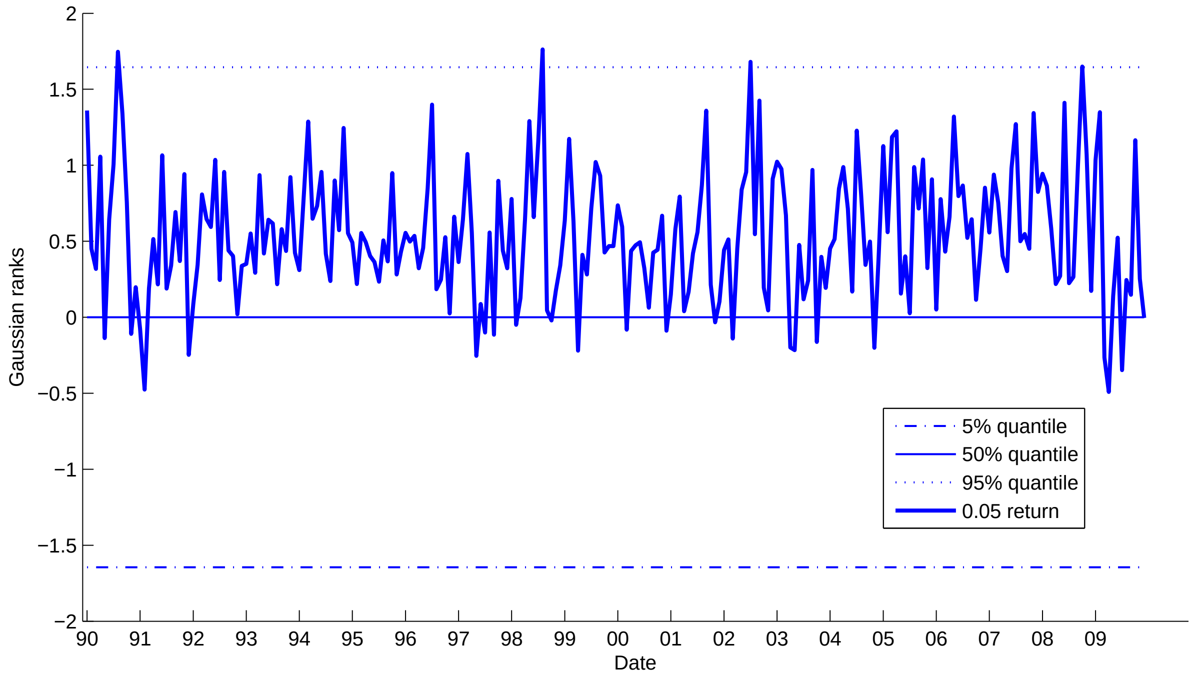
The figure displays the time series of cross-sectional distributions of monthly CRSP stock returns from January 1990 to December 2009. The monthly returns are computed as $y_{i,t} = p_{i,t}/p_{i,t-1} - 1$, where $p_{i,t}$ is the price of stock i at month t. The returns are not annualized and not in percentage. The CS probability density function (p.d.f.) are kernel estimates with Gaussian kernel and bandwidths selected by the rule of thumb in Silverman (1986).

Figure 2: Time series of quantiles of the CS distributions of monthly CRSP stock returns.



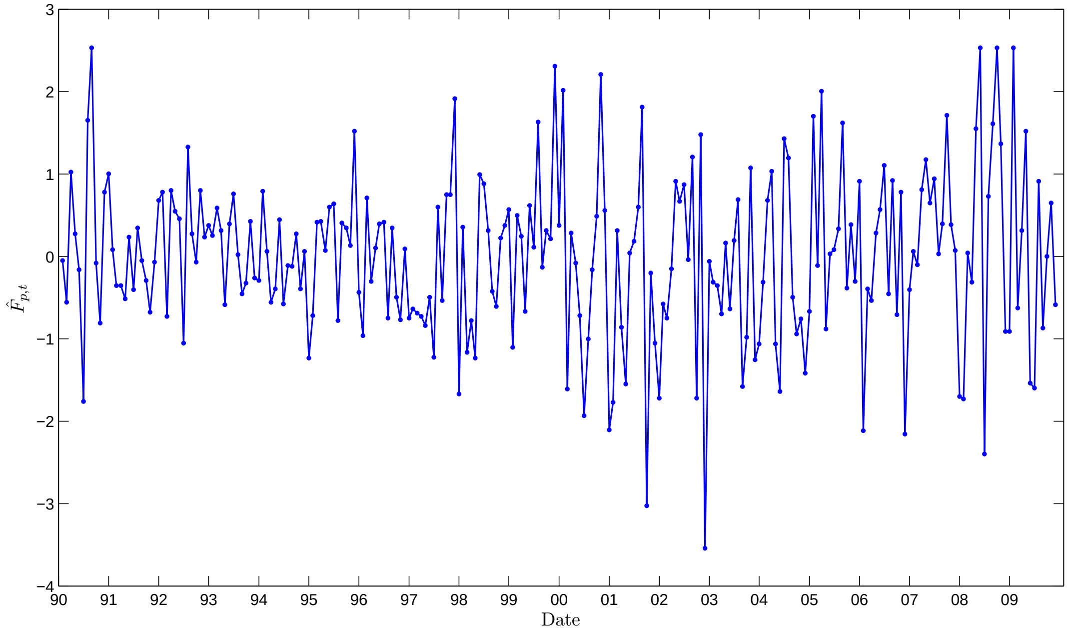
The figure displays the time series of the 5% CS quantile (lower dash-dotted line), the 25% CS quantile (lower solid line), the CS median (bold solid line), the 75% CS quantile (upper solid line), the 95% CS quantile (upper dash-dotted line).

Figure 3: Time series of ex-post Gaussian ranks associated with a constant monthly return of 0.05.



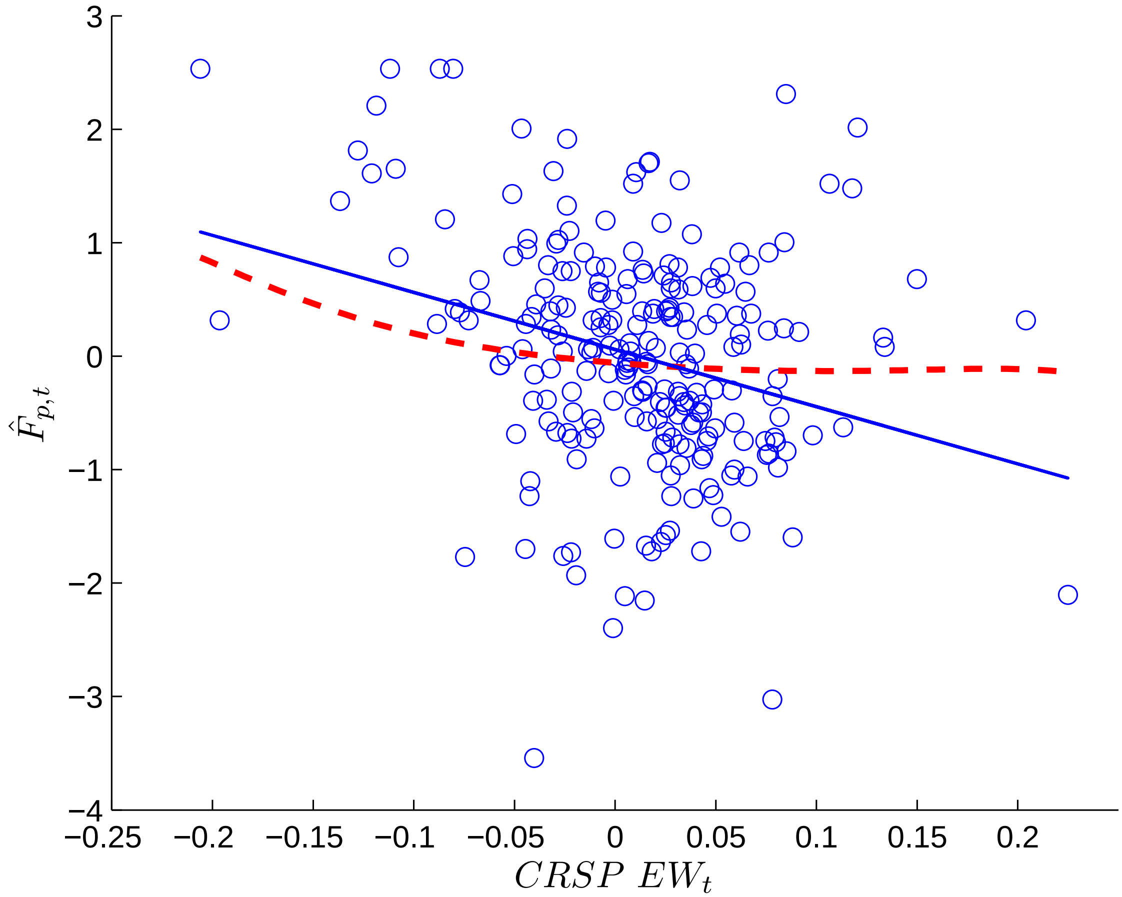
The solid bold line is the time series of ex-post Gaussian ranks of an asset with constant 0.05 monthly return. The dashed-dotted, thin solid and dotted horizontal lines represent the Gaussian ranks of a constant position at the 5%, 50% and 95% quantile of the cross-sectional distribution at each month, respectively.

Figure 4: Time series of positional factor estimates $\hat{F}_{p,t}$.



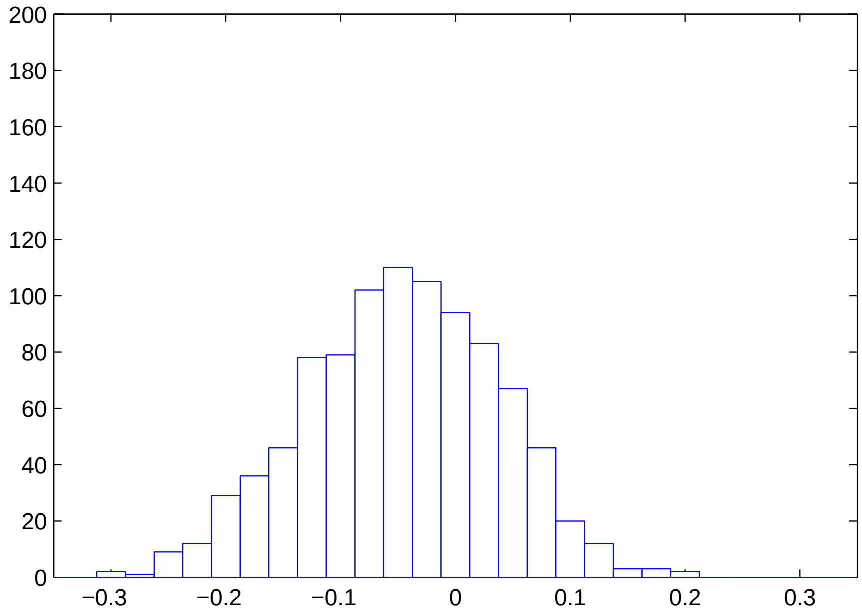
Monthly time series of estimates of the positional factor $\hat{F}_{p,t}$, obtained via the estimator in Equation (3.4), computed as described in Appendix A.4.

Figure 5: Positional factor vs. CRSP EW index returns.

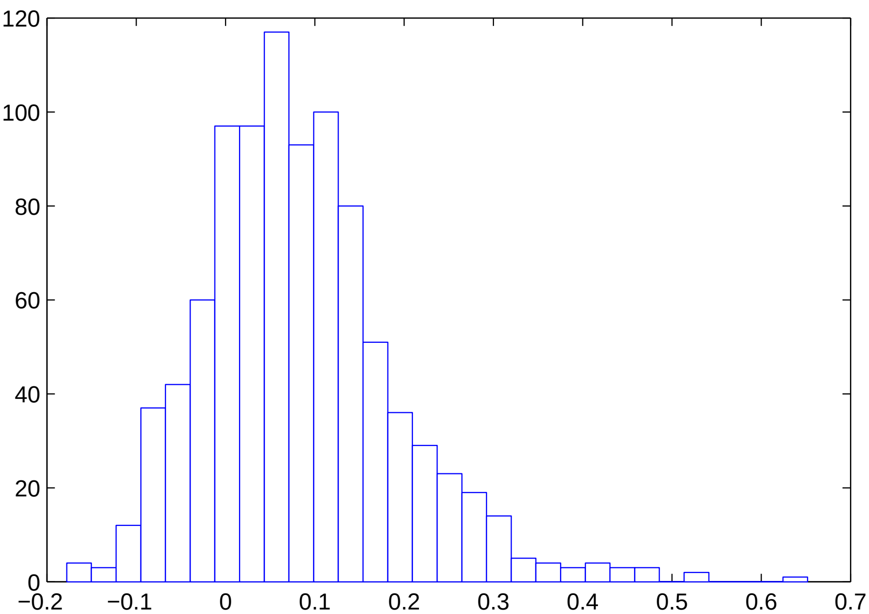


The figure displays a scatterplot of the estimates $\hat{F}_{p,t}$ of the positional factor versus the monthly returns of the equally weighted (EW) CRSP index. The solid and dashed lines correspond to a linear regression fit, and a nonparametric regression curve, respectively. The nonparametric regression curve is obtained as a kernel smoothing regression using a Gaussian kernel, with bandwidth equal to 0.0617, selected using the rule-of-thumb suggested by Bowman and Azzalini (1997).

Figure 6: Histograms of estimated individual effects.



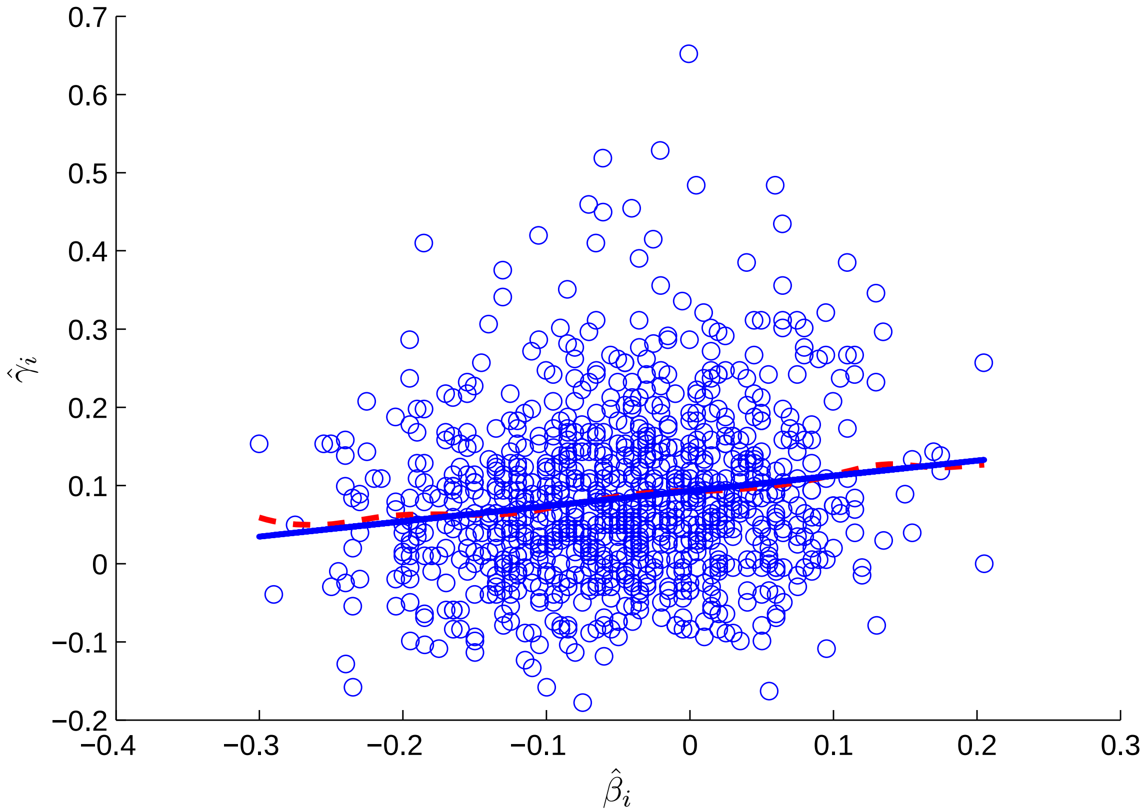
(a) $\hat{\beta}_i$



(b) $\hat{\gamma}_i$

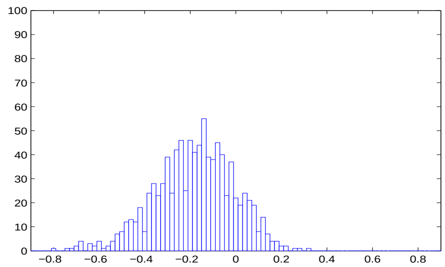
The figure displays the histograms of the estimated individual effects $\hat{\beta}_i$ and $\hat{\gamma}_i$ in panels (a) and (b), respectively. The estimates are obtained via the estimator in Equation (3.4), computed as described in Appendix A.4.

Figure 7: Scatterplot of $\hat{\gamma}_i$ vs. $\hat{\beta}_i$.

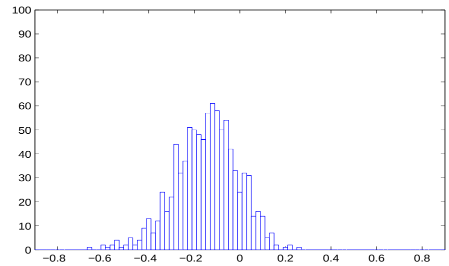


The figure displays the scatterplot of $\hat{\gamma}_i$ vs. $\hat{\beta}_i$, as well as the fitted linear regression line (solid) and the kernel smoothing regression line (dashed) corresponding to the regression of $\hat{\gamma}_i$ on $\hat{\beta}_i$. The smoothing regression is performed using a Gaussian kernel, with bandwidth equal to 0.0248, selected using the rule-of-thumb suggested by Bowman and Azzalini (1997).

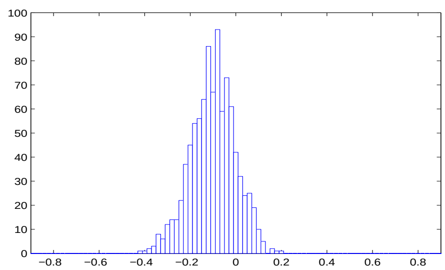
Figure 8: Histograms of positional persistence $\hat{\rho}_{i,t}$ as function of $F_{p,t}$.



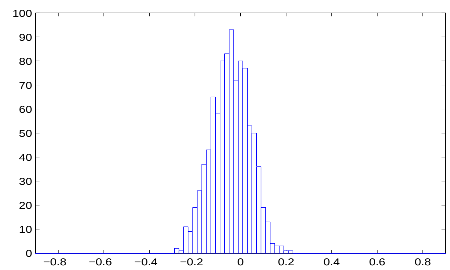
(a) $F_{p,t} = -1.7106$



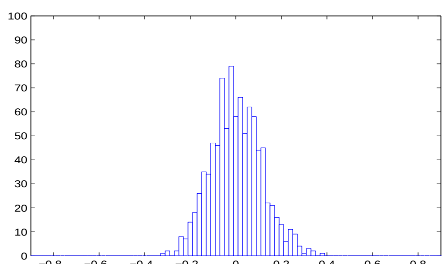
(b) $F_{p,t} = -1.2298$



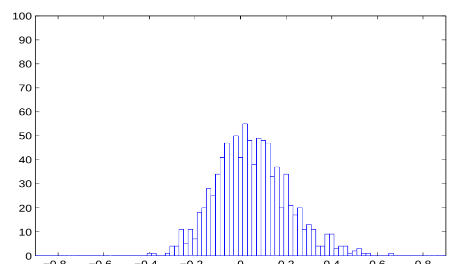
(c) $F_{p,t} = -0.6365$



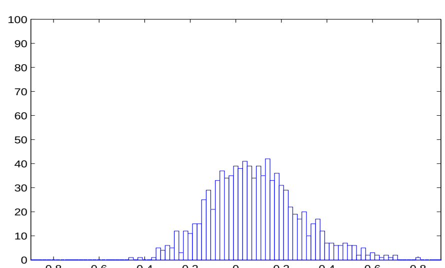
(d) $F_{p,t} = 0.0417$



(e) $F_{p,t} = 0.5985$



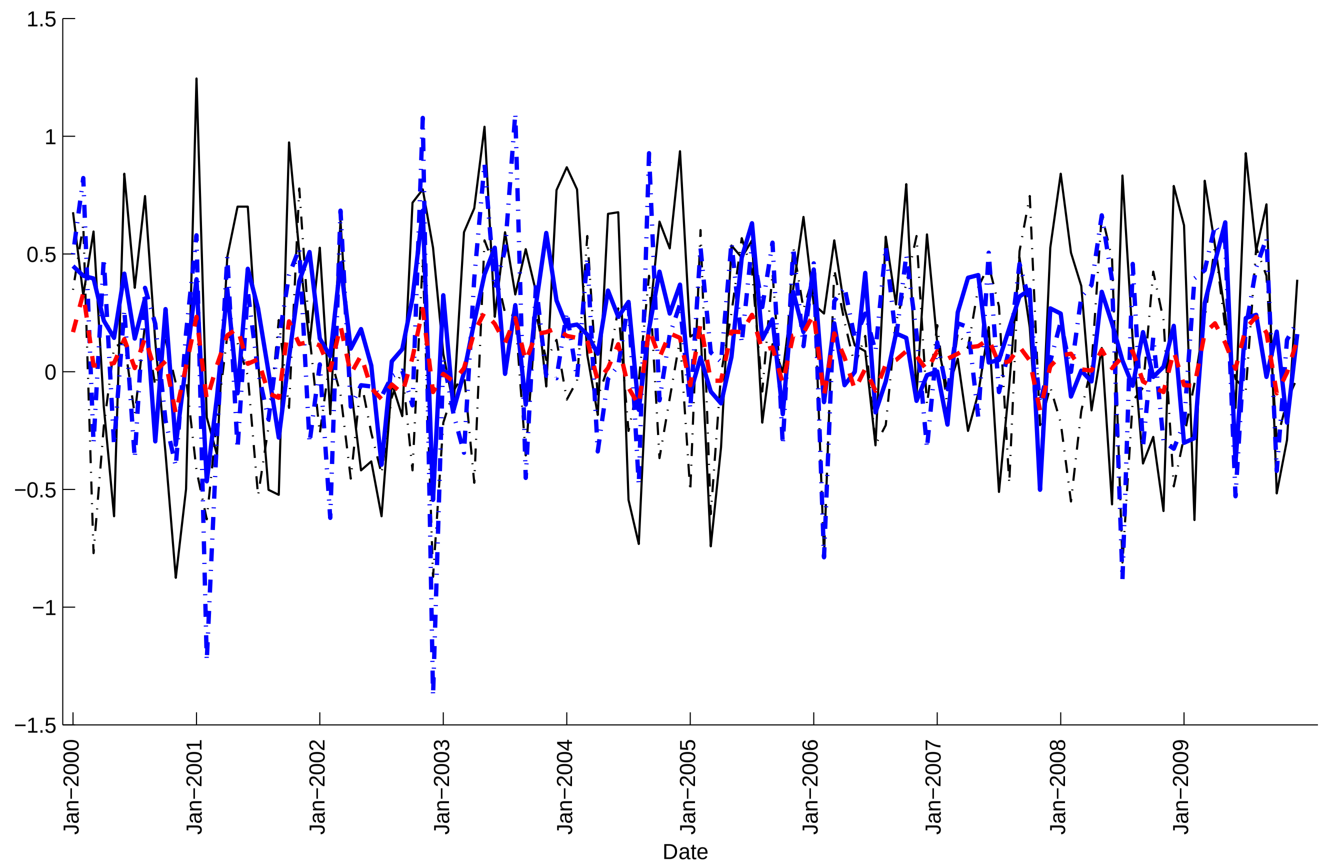
(f) $F_{p,t} = 1.2018$



(g) $F_{p,t} = 1.6792$

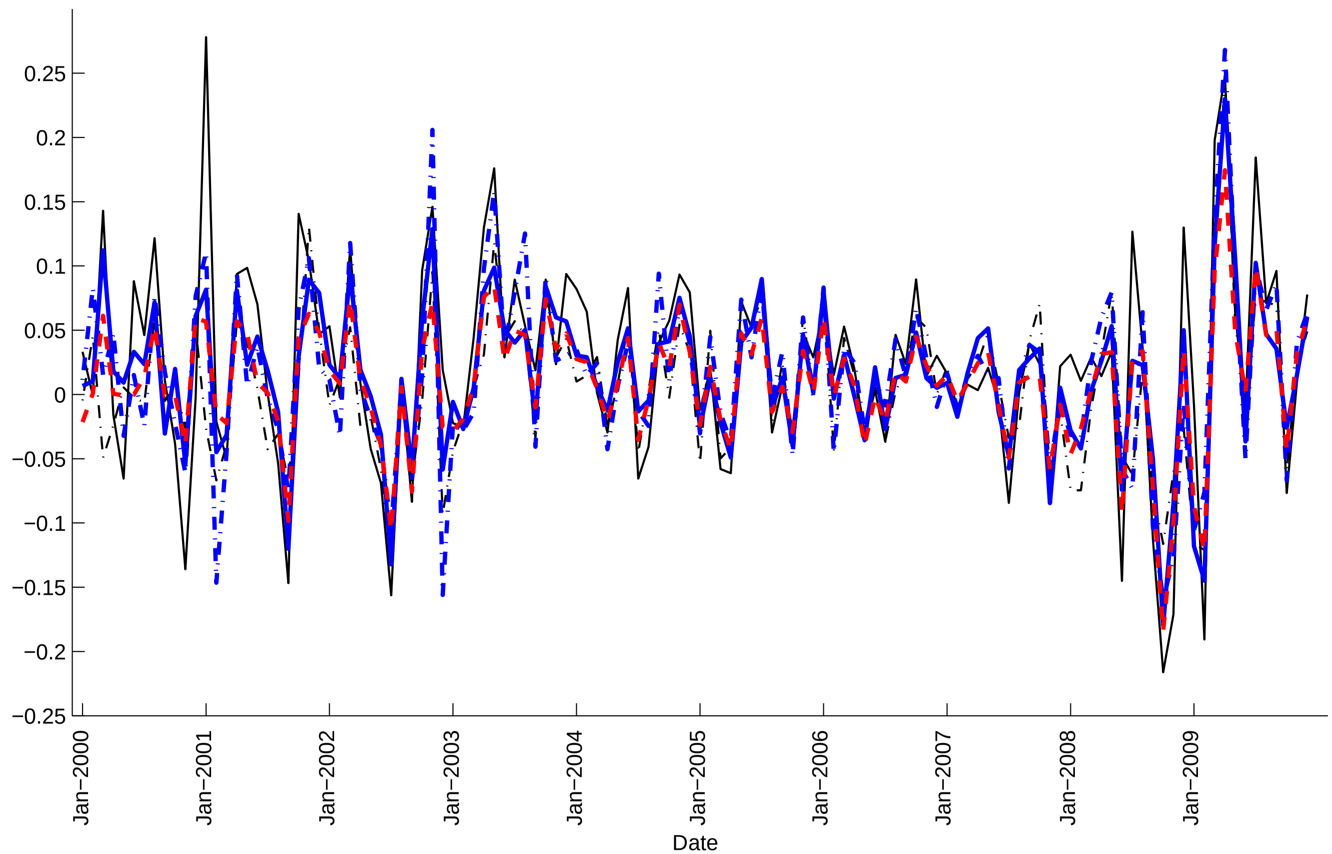
The figure displays the histograms of the positional persistence $\hat{\rho}_{i,t} = \Psi(\hat{\beta}_i + \hat{\gamma}_i F_{p,t})$ across stocks i, for different values of the positional factor $F_{p,t}$. These values of $F_{p,t}$ correspond to the 5%, 10%, 25%, 50%, 75%, 90% and 95% quantiles of the historical distribution of the positional factor in panels (a), (b), (c), (d), (e), (f) and (g), respectively.

Figure 9: Ex-post properties of the portfolio strategies, 2000-2009.



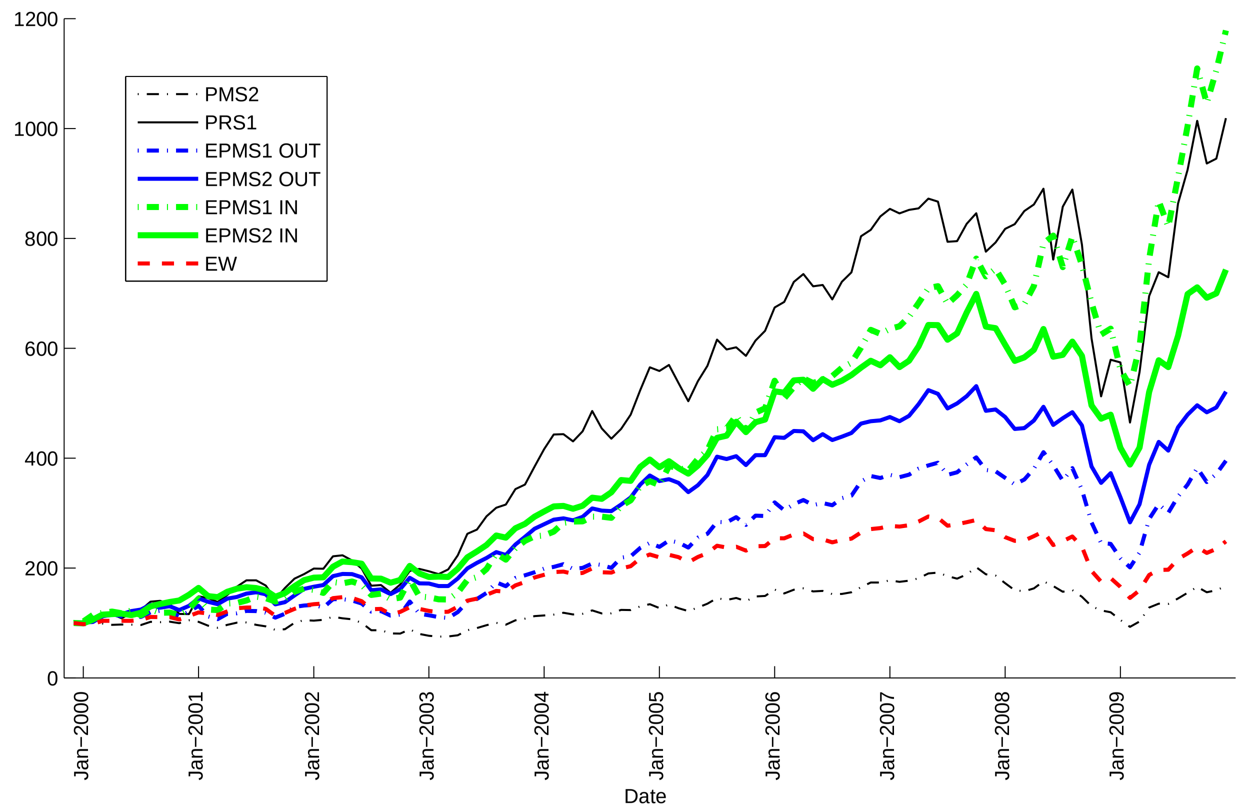
Panel (a): The figure displays the monthly time series of the Gaussian cross-sectional ranks of five portfolio strategies over the period from January 2000 to December 2009. The Gaussian ranks are computed w.r.t. the CS distribution of the monthly returns of all the NYSE, AMEX and NASDAQ stocks in our sample. The dashed-dotted thin black line corresponds to strategy PMS2, the solid thin black line corresponds to strategy PRS1, the dashed-dotted bold blue line corresponds to strategy EPMS1 OUT, the solid bold blue line corresponds to strategy EPMS2 OUT, and the dashed bold red line to the equally weighted portfolio. The strategies PMS1 and PRS2 perform worse than PMS2 and PRS1, respectively, in terms of the criteria in Table 1. For readability purpose, their series of Gaussian cross-sectional ranks are not displayed. The strategies are described in Section 4.2.

Figure 8: Ex-post properties of the portfolio strategies, 2000-2009.



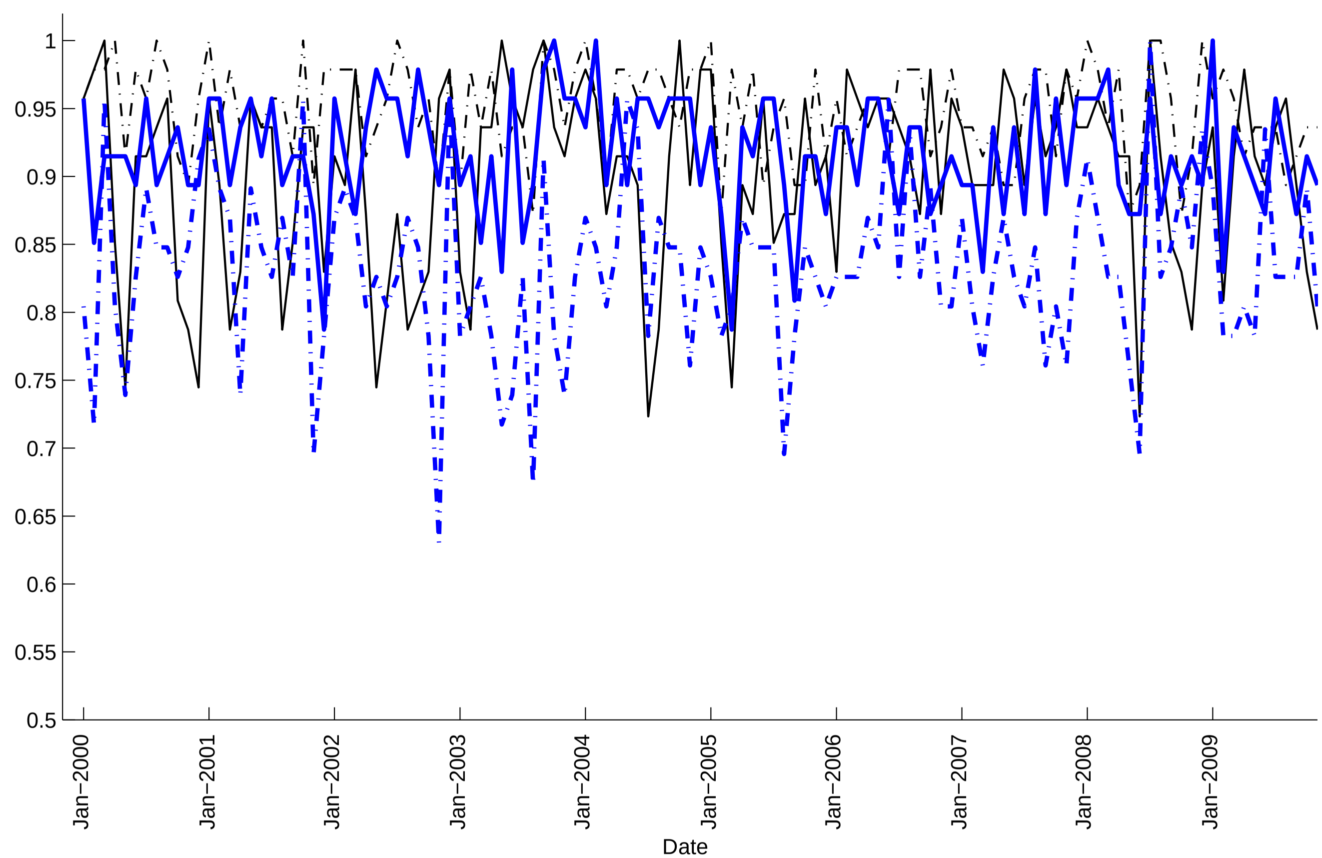
Panel (b): The figure displays the monthly time series of excess returns of five portfolio strategies over the period from January 2000 to December 2009. The dashed-dotted thin black line corresponds to strategy PMS2, the solid thin black line corresponds to strategy PRS1, the dashed-dotted bold blue line corresponds to strategy EPMS1 OUT, the solid bold blue line corresponds to strategy EPMS2 OUT, and the dashed bold red line to the equally weighted portfolio. The strategies PMS1 and PRS2 perform worse than PMS2 and PRS1, respectively, in terms of the criteria in Table 1. For readability purpose, their series of excess returns are not displayed. The strategies are described in Section 4.2.

Figure 8: Ex-post properties of the portfolio strategies, 2000-2009.



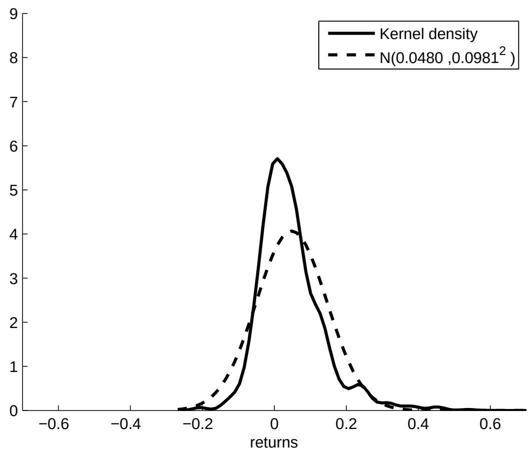
Panel (c): The figure displays the monthly time series of cumulated excess returns of five portfolio strategies over the period from January 2000 to December 2009. The dashed-dotted thin black line corresponds to strategy PMS2, the solid thin black line corresponds to strategy PRS1, the solid bold blue line corresponds to strategy EPMS2 OUT, the dashed-dotted bold blue line corresponds to strategy EPMS1 OUT, the dashed-dotted bold green line corresponds to strategy EPMS1 IN, the solid green line corresponds to strategy EPMS2 IN, and the dashed bold red line to the equally weighted portfolio. The strategies PMS1 and PRS2 perform worse than PMS2 and PRS1, respectively, in terms of the criteria in Table 1. For readability purpose, their series of cumulated returns are not displayed. The strategies are described in Section 4.2.

Figure 8: Ex-post properties of the portfolio strategies, 2000-2009.

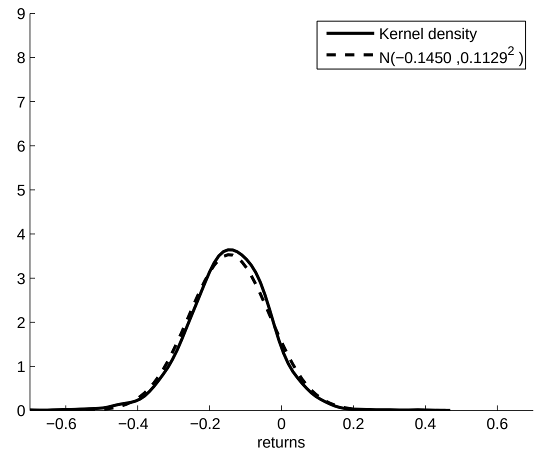


Panel (d): The figure displays the monthly time series of turnover of four portfolio strategies PMS2, PRS1, EPMS1, and EPMS2 over the period from January 2000 to December 2009. The dashed-dotted thin black line corresponds to strategy PMS2, the solid thin black line corresponds to strategy PRS1, the dashed-dotted bold blue line corresponds to strategy EPMS1 OUT, the solid bold blue line corresponds to strategy EPMS2 OUT. The turnover is computed as the proportion of selected stocks which are not kept in the portfolio between two consecutive dates. The strategies PMS1 and PRS2 perform worse than PMS2 and PRS1, respectively, in terms of the criteria in Table 1. For readability purpose, their turnover are not displayed. The strategies are described in Section 4.2.

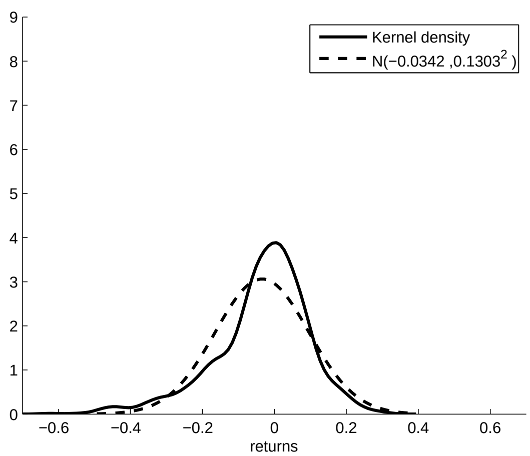
Figure 10: Cross-sectional distributions of monthly CRSP stock returns.



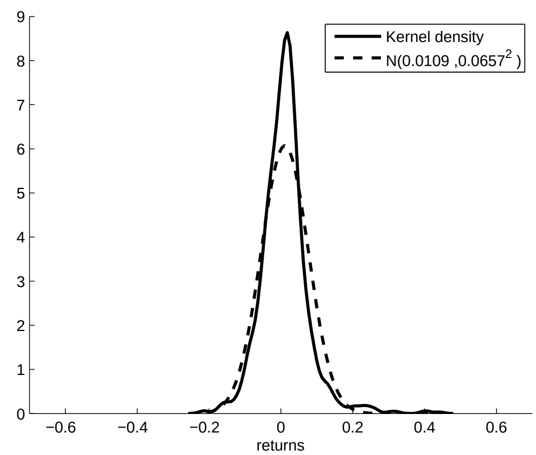
(a) Cross-sectional distribution on July 1995.



(b) Cross-sectional distribution on August 1998.



(c) Cross-sectional distribution on November 2000.

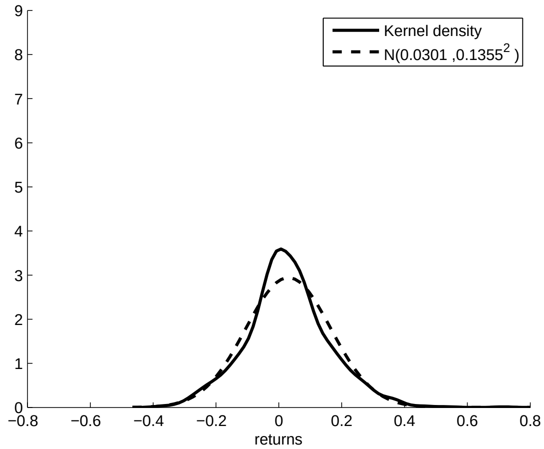


(d) Cross-sectional distribution on December 2006.

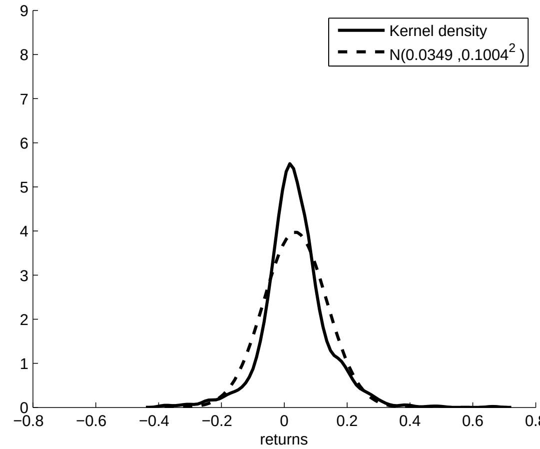
Each panel displays the kernel estimator of the CS density of the CRSP stock returns for a particular month (solid line), and compares it with a Gaussian distribution $N(\hat{\mu}_t, \hat{\sigma}_t^2)$ (dashed line), where $\hat{\mu}_t$ and $\hat{\sigma}_t^2$ are the cross-sectional mean and variance:

$$\hat{\mu}_t = \frac{1}{n}\sum_{i=1}^{n} y_{i,t}\,, \qquad \hat{\sigma}_t^2 = \frac{1}{n}\sum_{i=1}^{n}(y_{i,t} - \hat{\mu}_t)^2.$$

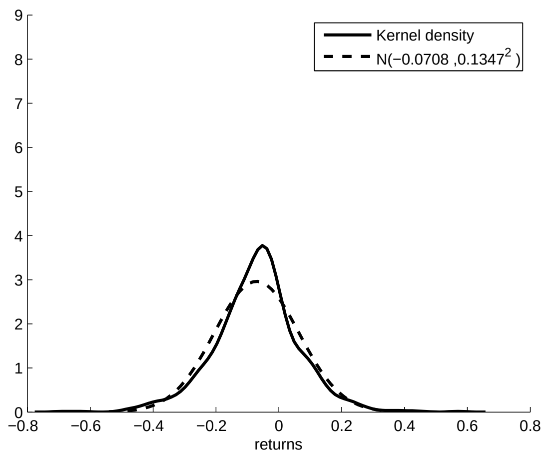
Figure 11: Cross-sectional distributions around the 2008 Lehman Brothers crisis.



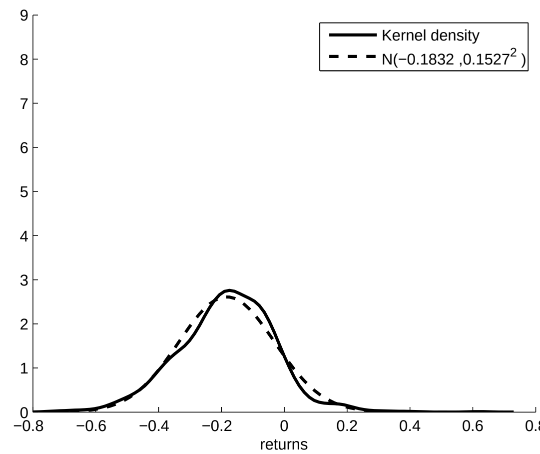
(a) Cross-sectional distribution on July 2008.



(b) Cross-sectional distribution on August 2008.



(c) Cross-sectional distribution on September 2008.

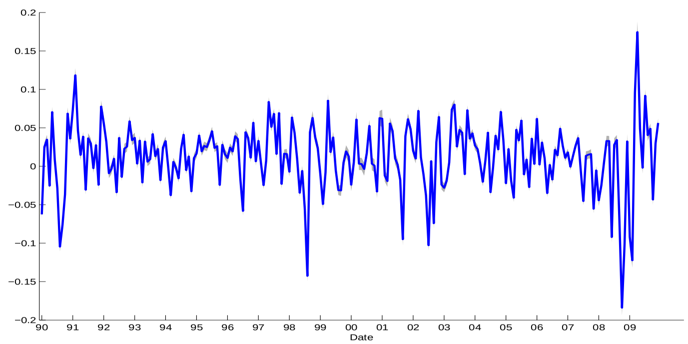


(d) Cross-sectional distribution on October 2008.

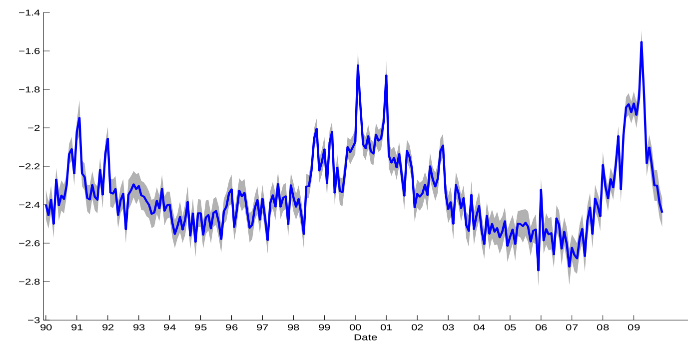
Each panel displays the kernel estimator of the CS density of the CRSP stock returns for a particular month (solid line), and compares it with a Gaussian distribution $N(\hat{\mu}_t, \hat{\sigma}_t^2)$ (dashed line), where $\hat{\mu}_t$ and $\hat{\sigma}_t^2$ are the cross-sectional mean and variance:

$$\hat{\mu}_t = \frac{1}{n}\sum_{i=1}^{n} y_{i,t}\,, \qquad \hat{\sigma}_t^2 = \frac{1}{n}\sum_{i=1}^{n}(y_{i,t} - \hat{\mu}_t)^2.$$

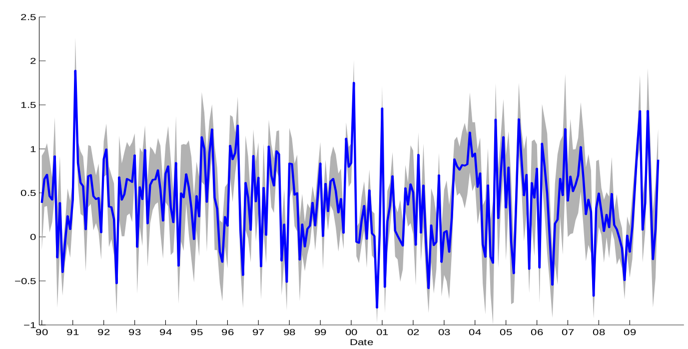
Figure 12: Time series of estimated distributional macro-factors.



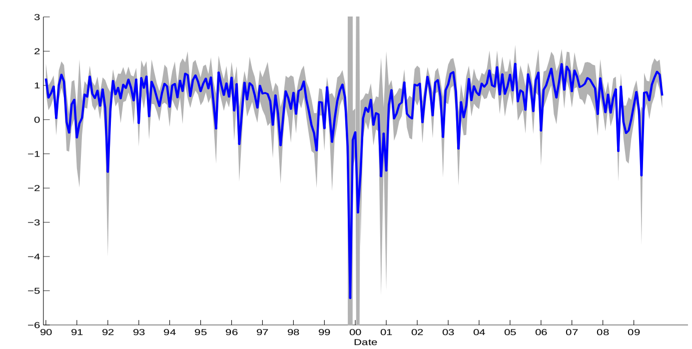
(a) CS mean ($\hat{\mu}_t$)



(b) log CS standard deviation ($\log \hat{\sigma}_t$)



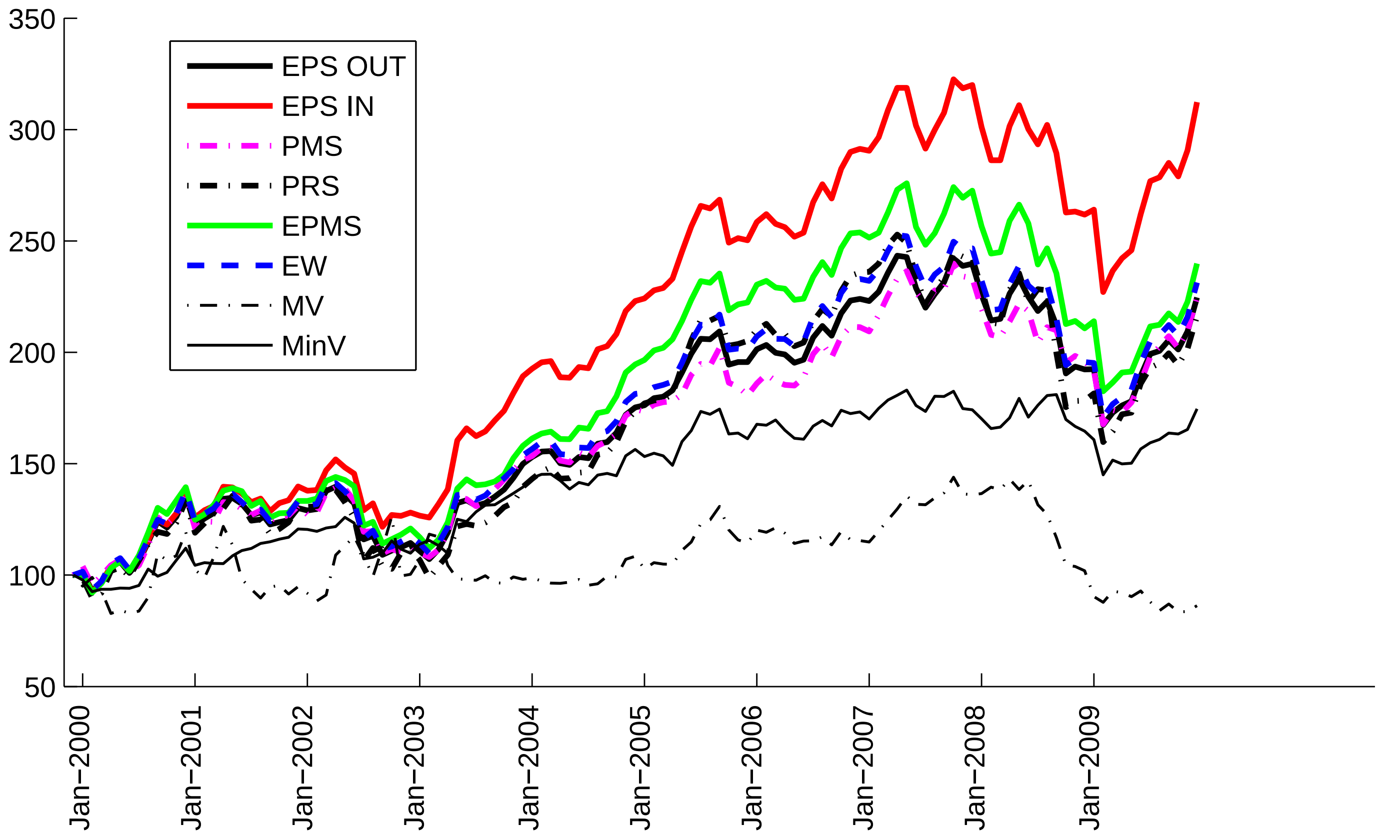
(c) CS skewness ($\hat{s}_t$)



(d) log CS excess kurtosis ($\log \hat{k}_t^*$)

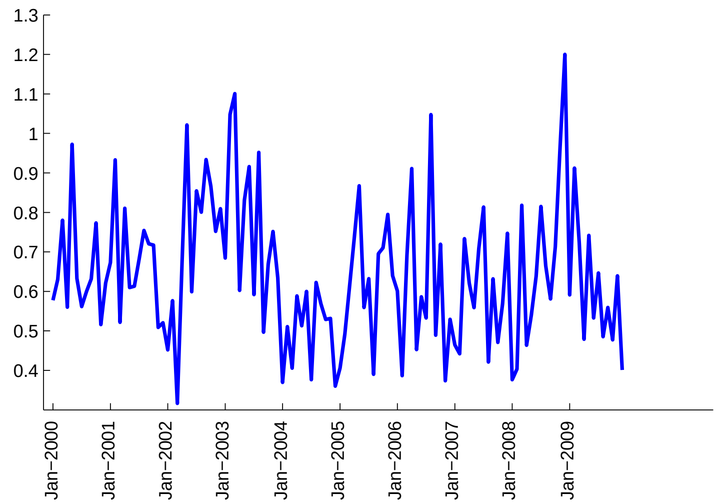
Panel (a) displays the time series of the CS mean $\hat{\mu}_t = \frac{1}{n}\sum_{i=1}^{n} y_{i,t}$ of the stock returns. Panel (b) displays the time series of the log CS standard deviation $\log \hat{\sigma}_t$, where $\hat{\sigma}_t^2 = \frac{1}{n}\sum_{i=1}^{n}(y_{i,t} - \hat{\mu}_t)^2$. Panel (c) displays the time series of the CS skewness $\hat{s}_t = \frac{1}{n}\sum_{i=1}^{n}(y_{i,t} - \hat{\mu}_t)^3/\hat{\sigma}_t^3$. Panel (d) displays the time series of the log CS excess kurtosis $\log \hat{k}_t^*$, where $\hat{k}_t^* = \hat{k}_t - 3(1 + \hat{s}_t^2/2)$, and $\hat{k}_t = \frac{1}{n}\sum_{i=1}^{n}(y_{i,t} - \hat{\mu}_t)^4/\hat{\sigma}_t^4$. The grey areas represent 95% pointwise confidence bands.

Figure 13: Time series of cumulative returns of the portfolio strategies in the utilities sector, 2000-2009.

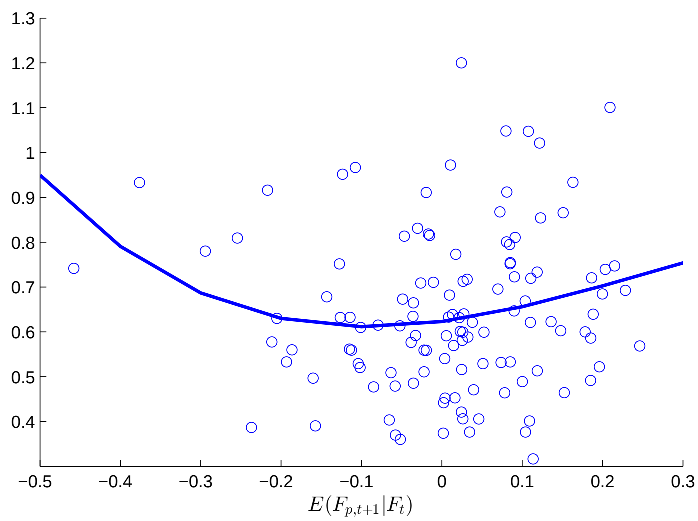


The Figure displays the cumulated excess returns of the eight portfolio strategies based on 57 stocks in the utilities sector over the period from January 2000 to December 2009. The solid bold black line corresponds to strategy EPS OUT, the solid bold red line corresponds to strategy EPS IN, the bold dash-dotted magenta line corresponds to strategy PMS, the bold dash-dotted black line corresponds to strategy PRS, the bold solid green line corresponds to strategy EPMS, the bold dashed blue line to the equally weighted portfolio, the thin solid black line corresponds to the minimum-variance strategy and the thin black dash-dotted line corresponds to the mean-variance portfolio.

Figure 14: Observed measure of relative discrepancy of optimal positional allocation from EW portfolio for the subsample of utilities.



(a) $n\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a^*_{i,t}-\frac{1}{n})^2}$



(b) $n\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a^*_{i,t}-\frac{1}{n})^2}$ vs. $E(F_{p,t+1}|F_t)$

Panel (a) displays the time series of the observed measure of relative discrepancy of the optimal positional allocation from the EW portfolio $n\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a^*_{i,t}-\frac{1}{n})^2}$ for the subsample of utilities. Panel (b) displays the scatterplot of $n\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a^*_{i,t}-\frac{1}{n})^2}$ vs. $E(F_{p,t+1}|F_t)$, as well as the 5-th order polynomial fit of the data (solid blue line). The conditional expectation $E(F_{p,t+1}|F_t)$ is computed by using the estimated VAR(1) model of the macro-factor process (see Section 5.2).

APPENDIX A

Appendix A.1: The factor model of stock returns and the ex-ante ranks

In this appendix we describe more formally the factor structure of stock returns introduced in Section 2, and give the definition of ex-ante ranks. We assume a factor structure for asset returns as in the assumption below.

Assumption A. 1. *i) The individual return histories $y_i = (y_{i,t})$, with $i = 1, \ldots, n$, are independent and identically distributed (i.i.d.) conditionally on the path of an unobservable factor (F_t).*

ii) The conditional distribution of the return $y_{i,t}$ given the past return history $\underline{y_{i,t-1}} = (y_{i,t-1}, y_{i,t-2}, ...)$ and the entire factor path (F_t) depends on the latter by means of the current and past factor values $\underline{F_t} = (F_t, F_{t-1}, ...)$ only.

The factor F_t can be multidimensional and corresponds to systematic, or common, risks. When the unobservable factor path (F_t) is integrated out, the individual asset returns histories become dependent. Under Assumption A.1 *i)*, the factor process (F_t) fully captures the dependence across assets returns. Assumption A.1 *ii)* implies that the conditional distribution of F_t given the past histories of the factor $\underline{F_{t-1}}$ and the returns $\underline{y_{i,t-1}}$, $i = 1, ..., n$, is independent of the latter, that is, the factor process is exogenous.

The unconditional distribution of assets returns is exchangeable, that is, invariant to asset permutations. This property corresponds to the ex-ante homogeneity of the population of assets. However, the assets are ex-post heterogeneous, as they have different distributions conditional on the past return histories. Indeed, under Assumption A.1, the model is compatible with assets having different individual unobservable characteristics (such as the factor sensitivities and idiosyncratic volatilities for stocks, or the manager's skill for fund portfolios) and the past return histories are informative for these individual unobservable characteristics.

Assumption A. 2. *The process (F_t) is strictly stationary and Markov.*

Under Assumption A.1, the returns at date t, that are $y_{1,t}, \dots, y_{n,t}$, are conditionally i.i.d. variables admitting a cumulative distribution function (c.d.f.) H_t^* defined by $H_t^*(y) = \mathbb{P}(y_{i,t} \leq y|\underline{F_t})$. The distribution H_t^* is conditional on the current and past realizations $\underline{F_t} = (F_t, F_{t-1}, \dots)$ of the systematic factor.

Assumption A. 3. *The cross-sectional returns c.d.f. H_t^* is continuous and strictly increasing.*

Under Assumption A.3, at any date t there is a one-to-one mapping between the stock returns and the ex-ante ranks, that are defined next.

Definition 1. *i) The uniform ex-ante ranks are defined as $u_{i,t}^* = H_t^*(y_{i,t})$.*

ii) The Gaussian ex-ante ranks are defined as $u_{i,t} = \Phi^{-1}(u_{i,t}^*) = H_t(y_{i,t})$, where Φ denotes the c.d.f. of the standard normal distribution, and $H_t = \Phi^{-1} \circ H_t^*$.

The ex-ante uniform ranks (resp. the ex-ante Gaussian ranks) at a given date are conditionally i.i.d. variables with cross-sectional uniform distribution on the interval $[0, 1]$ (resp. a standard Gaussian distribution).

The model introduced in Sections 2-5 can be cast in the framework of Assumptions A.1 - A.3 with multiple factor $F_t = (F'_{d,t}, F'_{p,t})'$. The specification is such that the CS distribution $H_t^*(\cdot)$ depends on the current value of component $F_{d,t}$, and belongs to the Variance-Gamma family (Section 5.1 and Appendix A.5). The component $F_{p,t}$ drives the positional persistence of the Gaussian ranks (Section 3.1). The unobservable characteristics $\delta_i = (\beta_i, \gamma_i)'$ introduce heterogeneity in the positional persistence of stocks (Section 3.1).

Appendix A.2: Positional management strategies

i) Derivation of the optimal positional allocation

In this section we derive the optimal positional allocation. For given budget w_r allocated in the risky assets, the future value of the risky part of the portfolio is $w_r + \gamma' y_{t+1} = w_r(\alpha' y_{t+1} + 1)$, where the dollar allocations vector γ (resp., the relative allocations vector α) is such that $\gamma' e = w_r$ (resp., $\alpha' e = 1$). The rank of this future portfolio value has to be computed with respect to the cross-sectional distribution of the values at month $t+1$ of portfolios with budget w_r invested at month t in any single risky asset i. These values are $w_r(y_{i,t+1}+1)$ and their cross-sectional distribution is:

$$\tilde{H}_{t+1}(w) = \mathbb{P}[w_r(y_{i,t+1}+1) \leq w|\underline{F_{t+1}}] = \mathbb{P}[y_{i,t+1} \leq w/w_r - 1|\underline{F_{t+1}}] = H^*_{t+1}(w/w_r - 1),$$

where H^*_{t+1} is the cross-sectional distribution of stock returns at month $t+1$. Thus, the Gaussian rank of the future value of the risky part of the portfolio with dollar allocation γ is given by:

$$\begin{aligned} \Phi^{-1}\left[\tilde{H}_{t+1}(w_r + \gamma' y_{t+1})\right] &= \Phi^{-1}\left[H^*_{t+1}(\gamma' y_{t+1}/w_r)\right] \\ &= H_{t+1}(\gamma' y_{t+1}/w_r) = H_{t+1}(\alpha' y_{t+1}). \end{aligned}$$

The optimal positional dollar allocation γ^*_t is obtained by maximizing the expected positional utility of the Gaussian rank of the future portfolio value subject to the budget constraint:

$$\gamma^*_t = \underset{\gamma:\ \gamma' e = w_r}{\arg\max}\ E_t\left[\mathscr{U}\left(H_{t+1}(\gamma' y_{t+1}/w_r)\right)\right].$$

The solution is $\gamma^*_t = w_r \alpha^*_t$, where the optimal positional relative allocation α^*_t is given in equation (2.4).

ii) Aggregation of ranks

In this section we discuss the criterion function for positional allocation in terms of aggregation of ranks. There exist two ways to aggregate ranks. *a)* Let us consider a set of weights $\pi_1, \dots, \pi_n$ with $\pi_i \geq 0$, for $i = 1, \dots, n$, and $\sum_{i=1}^{n} \pi_i = 1$. It is usual to aggregate ranks by considering either the quantity $\sum_{i=1}^{n} \pi_i u^*_{i,t}$, or the quantity $\sum_{i=1}^{n} \pi_i u_{i,t}$. This ad-hoc approach is frequently used, for instance for selecting the stocks to include in a market index with a given number of assets in order to account jointly for the capitalization of the last month, the capitalization of the last three months and different liquidity measures. It has also been suggested in the latest draft released by the Basel Committee on Banking Supervision (BCBS, 2013) to aggregate the scores for five categories of importance of risks: size, cross-jurisdictional activity, interconnectedness, substitutability/financial institution infrastructure and complexity. *b)* An alternative consists in considering the rank of the associated weighted returns:

$$H_t^*\left(\sum_{i=1}^{n} \pi_i y_{i,t}\right) = H_t^*\left(\sum_{i=1}^{n} \pi_i H_t^{*-1}(u_{i,t}^*)\right), \tag{A.1}$$

or:

$$H_t\left(\sum_{i=1}^{n} \pi_i y_{i,t}\right) = H_t\left(\sum_{i=1}^{n} \pi_i H_t^{-1}(u_{i,t})\right). \tag{A.2}$$

We get H_t^*- and H_t-means of the individual ranks, respectively, instead of the time independent arithmetic means used in the first approach.

The second definition is more appealing in our framework. Indeed the set of weights π_i, $i = 1, \ldots, n$ can be considered as a portfolio allocation, for instance a portfolio of stocks, or a fund of funds. The average return $\sum_{i=1}^{n} \pi_i y_{i,t}$ is the portfolio (resp. fund of funds) return and is used to rank the new portfolio among the initial assets, that are the basic stocks (resp. funds). Moreover, definitions (A.1) and (A.2) are easily extended to negative π_i, or to π_i which are not summing up to 1. This is not the case with the first definition, since $\sum_{i=1}^{n} \pi_i u_{i,t}^*$ might be outside the unit interval $[0,1]$ for negative weights for instance. It is seen that the second definition of rank aggregation corresponds to the criterion function in equation (2.5).

iii) Nash equilibrium with endogenous cross-sectional distribution

In this section we describe the Nash equilibrium for the positional asset allocation problem. The cross-sectional distribution that defines the position of the portfolio value of a given investor is no more considered as exogenous. Instead, this distribution becomes endogenous and is determined at the equilibrium by the portfolio allocations of the population of investors.

Let us consider a continuum of portfolio managers, indexed by j, with $j \in [0,1]$. To simplify the exposition, let us assume that these managers have the same budget, normalized to 1, but different positional risk aversion parameters $\mathscr{A}_j$. These parameters admit a distribution $Q(d\mathscr{A})$. Each manager j allocates her budget in a portfolio of risky assets, with allocation vector α_j and portfolio return $\alpha_j' y_t$. We denote by $G(dy_t)$ the (exogenous) conditional distribution of the vector of asset returns, where for expository purpose we omit the time index $t-1$ of the conditional information. The distribution G is defined by the joint model of the assets returns (see Assumptions A.1-A.2 in Appendix A.1).

When solving the asset allocation problem, a given manager j considers the expected positioning of her portfolio with respect to the distribution of the portfolio values of the other managers. In the Nash equilibrium, the portfolio allocations of the other managers are considered as given by manager j. Let us denote by $\mathscr{H}_j$ the cumulative distribution function of the portfolio values of the other managers. The portfolio allocation problem of manager j becomes:

$$\begin{aligned}\alpha_j^* &= \underset{\alpha:\alpha' e=1}{\arg\max}\, E_G\left[\mathscr{U}\left(\mathscr{H}_j(\alpha' y_t); \mathscr{A}_j\right)\right] \\ &= \underset{\alpha:\alpha' e=1}{\arg\max} \int \mathscr{U}\left(\mathscr{H}_j(\alpha' y_t); \mathscr{A}_j\right) G(dy_t). \end{aligned} \tag{A.3}$$

Let us further assume that all managers have the same prior on the portfolio strategies of their competitors. Since any manager is negligible compared to the totality of the other managers, this assumption implies that the distribution $\mathscr{H}_j = \mathscr{H}$ is independent of j. Then, the solution of the constrained maximization problem (A.3) is such that:

$$\alpha_j^* = \alpha^*(\mathscr{A}_j, G, \mathscr{H}), \text{ say,}$$

for some given function α^* of the positional risk aversion parameter, the asset returns distribution and the cross-sectional portfolio values distribution.

Let us now derive the Nash equilibrium condition.[12] The future portfolio value for manager j is $\alpha^*(\mathscr{A}_j, G, \mathscr{H})' y_t$. This portfolio value is stochastic due to its dependence on the positional risk aversion $\mathscr{A}_j$, with distribution Q, and on the asset returns vector y_t, with distribution G. Then, the distribution $\mathscr{V}$, say, of the portfolio value in the population of managers is:

$$\mathscr{V}(w) = \mathbb{P}_{Q,G}[\alpha^*(\mathscr{A}_j, G, \mathscr{H})' y_t \leq w] = \int 1\{\alpha^*(\mathscr{A}_j, G, \mathscr{H})' y_t \leq w\} Q(d\mathscr{A}_j) G(dy_t).$$

This distribution depends on Q, G and $\mathscr{H}$, and let us make this dependence explicit by writing $\mathscr{V}(\cdot; Q, G, \mathscr{H})$, say. Then, the Nash equilibrium condition requires that the prior distribution $\mathscr{H}$ corresponds to the distribution derived ex-post, that is:

$$\mathscr{H} = \mathscr{V}(\cdot; Q, G, \mathscr{H}). \tag{A.4}$$

The equilibrium distribution $\mathscr{H}^* = \mathscr{H}^*(Q, G)$ is obtained by solving the functional fixed-point equa-

[12] Our approach can be compared with the equilibrium defined for two periods and two fund managers in Goriaev, Palomino, and Prat (2001).

tion (A.4). The discussion of the existence and uniqueness of the solution is beyond the scope of this paper.

Appendix A.3: ANOVA on Gaussian ranks

In order to motivate empirically the dynamic model for the Gaussian ranks with time variation and individual heterogeneity in positional persistence introduced in Section 3, let us perform a descriptive analysis of the empirical Gaussian rank processes. We consider the two-way panel regression:

$$\hat{u}_{i,t} = a + b_i + c_t + e_{i,t}, \tag{A.5}$$

where the empirical Gaussian ranks are explained in terms of a constant a, individual specific effects b_i, time specific effects c_t and disturbances $e_{i,t}$. For identification purpose, we set $\sum_{i=1}^{n} b_i = \sum_{t=1}^{T} c_t = 0$. The importance of individual and time effects to explain cross-sectional and time series variation of the Gaussian ranks can be assessed by testing the null hypotheses $H_0^1 : \{b_i = 0 \text{ for all } i\}$, $H_0^2 : \{c_t = 0 \text{ for all } t\}$, and the joint hypothesis $H_0^3 : \{b_i = 0 \text{ and } c_t = 0 \text{ for all } i \text{ and } t\}$. The values of the Fisher statistics $\mathscr{F}$ for the three hypotheses are provided below along with their corresponding critical values $\mathscr{F}^*$ at 95% level.

Eq. (A.5)	H_0^1	H_0^2	H_0^3
$\mathscr{F}$	0.725	0.005	0.578
$\mathscr{F}^*$	1.077	1.155	1.069

The Fisher statistics fail to reject the three null hypotheses H_0^1, H_0^2 and H_0^3. This descriptive analysis suggests that the rank processes feature neither individual, nor time effects in their levels. The estimate of parameter a is 0.0014. The absence of time effects, and a small estimate of parameter a, were expected since the cross-sectional distribution of the Gaussian ranks $u_{i,t}$ is standard Gaussian at every date t.

In order to test for individual and time effects in positional persistence, we next consider the regression:

$$(\hat{u}_{i,t} - \bar{\hat{u}}_{i,\cdot})(\hat{u}_{i,t-1} - \bar{\hat{u}}_{i,\cdot,-1}) = a + b_i + c_t + e_{i,t}, \tag{A.6}$$

where $\bar{\hat{u}}_{i,\cdot} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{i,t}$ is the time average of the Gaussian ranks of stock i, and similarly for $\bar{\hat{u}}_{i,\cdot,-1}$. The explained variable in this regression is the cross-product of demeaned individual ranks at consecutive dates. We test the three hypotheses H_0^1, H_0^2 and H_0^3. The results for the test statistics are displayed next and show the presence of both individual and time effects in positional persistence.

Eq. (A.6)	H_0^1	H_0^2	H_0^3
$\mathscr{F}$	1.433	6.088	2.375
$\mathscr{F}^*$	1.077	1.155	1.069

Thus, in Section 3 we focus on the modelling of the positional persistence parameters.

Appendix A.4: The dynamics of ranks

i) Strict stationarity of the rank processes

Let us consider the rank dynamics in equations (3.1) and (3.2), and assume that the common factor $(F_{p,t})$ is a strictly stationary process (see Assumption A.2). Then, for any asset i, the rank process $(u_{i,t})$ is strictly stationary. Indeed, conditionally on any value $\delta_i = (\beta_i, \gamma_i)'$ of the random individual effect, the strict stationarity condition for a stochastic autoregressive process [see e.g. Bougerol and Picard (1992)], namely: $E\left[\log|\rho_{i,t}| \;\; |\, \delta_i\right] < 0$, is satisfied.

ii) Cross-sectional distribution of the ranks

Let us now verify that the cross-sectional distribution of the Gaussian ranks $u_{i,t}$, for i varying at date t, implied by equations (3.1) and (3.2) is standard Gaussian. By solving backward the autoregressive equation (3.1), we get an infinite-order Moving Average $MA(\infty)$ representation for process $u_{i,t}$, that is,

$$u_{i,t} = \sum_{\ell=0}^{\infty}\pi_{i,t}(\ell)\varepsilon_{i,t-\ell},$$

where the moving average coefficients $\pi_{i,t}(0) = \rho_{i,t}$ and $\pi_{i,t}(\ell) = \rho_{i,t}\rho_{i,t-1}...\rho_{i,t-\ell+1}\sqrt{1-\rho_{i,t-\ell}^2}$, for $\ell \geq 1$, are time-varying and stock-specific. Since the disturbances $(\varepsilon_{i,t})$ are independent Gaussian white noises and $\sum_{\ell=0}^{\infty} \pi_{i,t}(\ell)^2 = 1$, we get that variable $u_{i,t}$ admits a standard Gaussian $N(0,1)$ distribution conditional on the factor path (F_t) and individual heterogeneity δ_i. This implies that $u_{i,t}$ admits a standard Gaussian distribution conditional on the factor path only.

iii) A simple sequential updating procedure for numerical computation of the fixed effects estimators in equations (3.4)-(3.5)

Let us now provide an algorithm with a small degree of numerical complexity for computing of the fixed effects estimates of the factor values $F_{p,t}$, for $t = 1, ..., T$, and the individual effects β_i and γ_i for $i = 1, ..., n$ defined in equations (3.4)-(3.5). The Lagrangian function of the constrained maximization problem is:

$$\mathcal{L} = \sum_{t=1}^{T} \sum_{i=1}^{n} \phi(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) - \lambda \sum_{t=1}^{T} F_{p,t} - \mu \sum_{t=1}^{T} F_{p,t}^2,$$

where:

$$\phi(z, w; \rho) = -\frac{1}{2} \log(1-\rho^2) - \frac{(z-\rho w)^2}{2(1-\rho^2)},$$

$\rho_{i,t} = \Psi(\beta_i + \gamma_i F_{p,t}) = \Psi(\delta_i' x_t)$, with $\delta_i = (\beta_i, \gamma_i)'$ and $x_t = (1, F_{p,t})'$, as in equation (3.2), and λ and μ are the Lagrange multipliers for the constraints in (3.5). The first-order conditions for $F_{p,t}$, $t = 1, ..., T$ and δ_i, $i = 1, ..., n$ are given by:

$$\sum_{i=1}^{n} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} \gamma_i - \lambda - 2\mu F_{p,t} = 0, \quad t = 1, ..., T, \tag{A.7}$$

$$\sum_{t=1}^{T} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} x_t = 0, \quad i = 1, ..., n, \tag{A.8}$$

respectively, where $\psi_{i,t} = \Psi'(\beta_i + \gamma_i F_{p,t})$ and the partial derivative of the function ϕ w.r.t. ρ is given by:

$$\frac{\partial \phi}{\partial \rho}(z, w; \rho) = \frac{1}{1-\rho^2} \left\{ (z-\rho w) w - \rho \left[\frac{(z-\rho w)^2}{1-\rho^2} - 1 \right] \right\}.$$

By summing the equations in (A.7) over $t = 1, ..., T$, we get:

$$\sum_{i=1}^{n} \sum_{t=1}^{T} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} \gamma_i - T\lambda - 2\mu \sum_{t=1}^{T} F_{p,t} = 0.$$

The first term (resp. the third term) in the equation is equal to 0 from (A.8) [resp. from (3.5)]. It follows that $\lambda = 0$. Similarly, by multiplying both sides of equation (A.7) by $F_{p,t}$ and summing again over $t = 1, ..., T$, we get:

$$\sum_{i=1}^{n} \sum_{t=1}^{T} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} F_{p,t} \gamma_i - 2\mu \sum_{t=1}^{T} F_{p,t}^2 = 0.$$

The first term in the equation is equal to 0 from (A.8), while we have $\sum_{t=1}^{T} F_{p,t}^2 = T$ from (A.8). It follows that $\mu = 0$. The Lagrange multipliers are zero since the maximized function value is the same with or without the constraints (3.5). Thus, the estimators can be computed from the equations:

$$\sum_{i=1}^{n} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} \gamma_i = 0, \quad t = 1, ..., T, \tag{A.9}$$

$$\sum_{t=1}^{T} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}) \psi_{i,t} x_t = 0, \quad i = 1, ..., n, \tag{A.10}$$

imposing the identification constraints (3.5). We solve the system of equations (A.9) - (A.10) by a Newton-Raphson method, in which the updating is performed sequentially with respect to time and individual effects. In contrast to the joint updating, which would require the inversion of matrices of dimension $(2n + T, 2n + T)$ and has a large degree of numerical complexity, the sequential updating simplifies considerably the computation, since it allows to update the values of the effects $F_{p,t}$, and δ_i independently across dates and individuals without matrix inversions. Specifically, let $F_{p,t}^{(q)}$, $\delta_i^{(q)}$ denote the values of the parameters at step q satisfying the constraints (3.5), and let $x_t^{(q)}$, $\rho_{i,t}^{(q)}$ and $\psi_{i,t}^{(q)}$ be the corresponding values of x_t, $\rho_{i,t}$ and $\psi_{i,t}$. Let us expand equation (A.9) for date t w.r.t. $F_{p,t}$ around the solution at step q. We have:

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)}) \psi_{i,t}^{(q)} \gamma_i^{(q)} \\
&+ \left[\sum_{i=1}^{n} \left(\frac{\partial^2 \phi}{\partial \rho^2}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)}) [\psi_{i,t}^{(q)}]^2 [\gamma_i^{(q)}]^2 + \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)}) \tau_{i,t}^{(q)} [\gamma_i^{(q)}]^2 \right) \right] (F_{p,t} - F_{p,t}^{(q)}) \simeq 0,
\end{aligned}$$

where $\tau_{i,t}^{(q)} = \Psi''(\beta_i^{(q)} + \gamma_i^{(q)} F_{p,t}^{(q)})$. By solving the above approximate equation, the new values of the time effects up to an additive constant and a multiplicative scale are given by:

$$\begin{aligned} \tilde{F}_{p,t}^{(q+1)} &= F_{p,t}^{(q)} - \left[\sum_{i=1}^{n} \left(\frac{\partial^2 \phi}{\partial \rho^2}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)})[\psi_{i,t}^{(q)}]^2 + \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)})\tau_{i,t}^{(q)} \right) [\gamma_i^{(q)}]^2 \right]^{-1} \\ &\quad \cdot \left[\sum_{i=1}^{n} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q)})\psi_{i,t}^{(q)}\gamma_i^{(q)} \right]. \end{aligned} \tag{A.11}$$

Similarly, we update at step $q+1$ the individual effects by performing a Taylor expansion of the equations in (A.10) w.r.t. the β_i and γ_i individual by individual, by taking into account the update of the time effects at step $q+1$:

$$\begin{aligned} \tilde{\delta}_i^{(q+1)} &= \delta_i^{(q)} - \left[\sum_{t=1}^{T} \left(\frac{\partial^2 \phi}{\partial \rho^2}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q+1/2)})[\psi_{i,t}^{(q+1/2)}]^2 \right.\right. \\ &\qquad \left.\left. + \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q+1/2)})\tau_{i,t}^{(q+1/2)} \right) x_t^{(q+1)}[x_t^{(q+1)}]' \right]^{-1} \\ &\quad \times \left[\sum_{t=1}^{T} \frac{\partial \phi}{\partial \rho}(\hat{u}_{i,t}, \hat{u}_{i,t-1}; \rho_{i,t}^{(q+1/2)})\psi_{i,t}^{(q+1/2)} x_t^{(q+1)} \right], \end{aligned} \tag{A.12}$$

where $x_t^{q+1} = (1, \tilde{F}_{p,t}^{(q+1)})'$, $\rho_{i,t}^{(q+1/2)} = \Psi(\beta_i^{(q)} + \gamma_i^{(q)} \tilde{F}_{p,t}^{(q+1)})$ and similarly for $\psi_{i,t}^{(q+1/2)}$ and $\tau_{i,t}^{(q+1/2)}$. Finally, we get the estimates at step $q+1$ by recentering and rescaling the values in (A.11) - (A.12) to account for the constraints:

$$\begin{aligned} \hat{F}_{p,t}^{(q+1)} &= \frac{\tilde{F}_{p,t}^{(q+1)} - \frac{1}{T}\sum_{t=1}^{T} \tilde{F}_{p,t}^{(q+1)}}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\left[\tilde{F}_{p,t}^{(q+1)}\right]^2 - \frac{1}{T}\sum_{t=1}^{T}\tilde{F}_{p,t}^{(q+1)} \right)^2}}, \quad t = 1, ..., T, \\ \hat{\gamma}_i^{(q+1)} &= \tilde{\gamma}_i^{(q+1)} \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\left[\tilde{F}_{p,t}^{(q+1)}\right]^2 - \frac{1}{T}\sum_{t=1}^{T}\tilde{F}_{p,t}^{(q+1)} \right)^2}, \quad i = 1, ..., n, \\ \hat{\beta}_i^{(q+1)} &= \tilde{\beta}_i^{(q+1)} + \tilde{\gamma}_i^{(q+1)} \frac{1}{T}\sum_{t=1}^{T} \tilde{F}_{p,t}^{(q+1)}, \quad i = 1, ..., n. \end{aligned}$$

iv) Proof of equation (3.6)

From the assumption of Gaussian CS distribution, the future position of the risky portfolio return is [see equation (2.8)]:

$$\begin{aligned} H_{t+1}(\alpha' y_{t+1}) &= \alpha' u_{t+1} \\ &= \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \sum_{i=1}^{n} \alpha_i \sqrt{1-\rho_{i,t+1}^2}\varepsilon_{i,t+1}. \end{aligned} \tag{A.13}$$

Then, by using that the $(\varepsilon_{i,t})$ are independent Gaussian white noise processes, the expected positional utility is:

$$\begin{aligned} & -E\left[\exp\left(-\mathscr{A} H_{t+1}(\alpha' y_{t+1})\right) | \underline{F_t}, \underline{y_t}\right] \\ = & -E\left\{E\left[\exp\left(-\mathscr{A} H_{t+1}(\alpha' y_{t+1})\right) | \underline{F_{t+1}}, \underline{y_t}\right] | \underline{F_t}, \underline{y_t}\right\} \\ = & -E\left[\exp\left(-\mathscr{A} \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \frac{1}{2}\mathscr{A}^2 \sum_{i=1}^{n} \alpha_i^2 (1-\rho_{i,t+1}^2)\right) | \underline{F_t}, \underline{y_t}\right]. \end{aligned}$$

The Lagrangian function for the maximization of the expected positional utility w.r.t. the portfolio allocation α subject to the constraint $\alpha' e = 1$ is:

$$\mathcal{L} = -E\left[\exp\left(-\mathscr{A} \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \frac{1}{2}\mathscr{A}^2 \sum_{i=1}^{n} \alpha_i^2 (1-\rho_{i,t+1}^2)\right) | \underline{F_t}, \underline{y_t}\right] + \lambda(\alpha' e - 1),$$

where λ is the Lagrange multiplier. The first-order condition for α_i is:

$$\begin{aligned} & \mathscr{A} E\left[\left(\rho_{i,t+1} u_{i,t} - \mathscr{A}(1-\rho_{i,t+1}^2)\alpha_i\right) \exp\left(-\mathscr{A} \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \frac{1}{2}\mathscr{A}^2 \sum_{i=1}^{n} \alpha_i^2 (1-\rho_{i,t+1}^2)\right) | \underline{F_t}, \underline{y_t}\right] \\ & +\lambda = 0. \end{aligned}$$

In the conditional expectation, the exponential function can be replaced by $\exp\left(-\mathscr{A} H_{t+1}(\alpha' y_{t+1})\right)$. We deduce that the solution $\alpha = \alpha_t^*$ satisfies the implicit equation:

$$\alpha_{i,t}^* = \frac{1}{\mathscr{A}^2} \frac{\lambda}{E_t^{\alpha}(1-\rho_{i,t+1}^2)} + \frac{1}{\mathscr{A}}\xi_{i,t}, \tag{A.14}$$

where $\xi_{i,t}$ is defined in equation (3.8). From the constraint $\sum_{i=1}^{n} \alpha_{i,t}^* = 1$, we get that the Lagrange multiplier is such that:

$$\frac{\lambda}{\mathscr{A}^2} = \frac{1}{\sum_{i=1}^{n} [E_t^{\alpha}(1-\rho_{i,t+1}^2)]^{-1}} \left(1 - \frac{1}{\mathscr{A}} \sum_{i=1}^{n} \xi_{i,t}\right).$$

By replacing this expression into equation (A.14), we get equation (3.6).

v) Portfolio with least risky future rank

In this subsection, we assume that positional persistence is time invariant, i.e. $\rho_{i,t} = \rho_i$. Then, from (A.13) the conditional variance of the future portfolio rank given the individual assets ranks is:

$$V\left[H_{t+1}(\alpha' y_{t+1}) \mid u_t\right] \quad = \quad \sum_{i=1}^{n} \alpha_i^2 (1-\rho_i^2).$$

By maximizing this conditional variance w.r.t. α subject to the constraint $\alpha' e = 1$, we get the portfolio allocation with conditionally least risky future rank:

$$\alpha_i \quad = \quad \frac{(1-\rho_i^2)^{-1}}{\sum_{i=1}^{n} (1-\rho_i^2)^{-1}}.$$

vi) Proof of equation (3.9)

At first-order in the persistence parameter we have:

$$E_t^{\alpha}(1-\rho_{i,t+1}^2) \simeq 1,$$

and:

$$\begin{aligned} E_t^{\alpha}(\rho_{i,t+1}) \quad &= \quad \frac{E\left[\rho_{i,t+1} \exp\left(-\mathscr{A} \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \frac{1}{2} \mathscr{A}^2 \sum_{i=1}^{n} \alpha_i^2 (1-\rho_{i,t+1}^2)\right) |\underline{F_t}, \underline{y_t}\right]}{E\left[\exp\left(-\mathscr{A} \sum_{i=1}^{n} \alpha_i \rho_{i,t+1} u_{i,t} + \frac{1}{2} \mathscr{A}^2 \sum_{i=1}^{n} \alpha_i^2 (1-\rho_{i,t+1}^2)\right) |\underline{F_t}, \underline{y_t}\right]} \\ &\simeq \quad E\left[\rho_{i,t+1} | \underline{F_t}, \underline{y_t}\right] \simeq E\left[\rho_{i,t+1} | F_t, \delta_i\right]. \end{aligned}$$

Then, from (3.7) and (3.8) we get $w_{i,t} \simeq 1/n$ and $\xi_{i,t} \simeq E_t(\rho_{i,t+1})u_{i,t}$. By plugging these approximations in (3.6), we get equation (3.9).

vii) Proof of approximation (6.1)

From equation (3.9) we have:

$$\frac{1}{n}\sum_{i=1}^{n}(\alpha_{i,t}^{*} - 1/n)^2 = \frac{1}{\mathscr{A}^2}\frac{1}{n}\sum_{i=1}^{n}\left(E_t(\rho_{i,t+1})u_{i,t} - \frac{1}{n}\sum_{i=1}^{n}E_t(\rho_{i,t+1})u_{i,t}\right)^2. \tag{A.15}$$

When $n \to \infty$, the strong Law of Large Numbers implies:

$$\frac{1}{n}\sum_{i=1}^{n}\left(E_t(\rho_{i,t+1})u_{i,t} - \frac{1}{n}\sum_{i=1}^{n}E_t(\rho_{i,t+1})u_{i,t}\right)^2 \quad \to \quad V(E_t(\rho_{i,t+1})u_{i,t}|\underline{F_t}), \tag{A.16}$$

where the convergence is almost surely. Moreover, we have:

$$V(E_t(\rho_{i,t+1})u_{i,t}|\underline{F_t}) = E(E_t(\rho_{i,t+1})^2 u_{i,t}^2|\underline{F_t}) = E(E_t(\rho_{i,t+1})^2|F_t), \tag{A.17}$$

where we use that the Gaussian ranks are such that $u_{i,t} \sim N(0,1)$ conditional on $\underline{F_t}, \delta_i$, and the process of the macro-factors is Markov. Let us now approximate the quantity $E(E_t(\rho_{i,t+1})^2|F_t)$ for small positional persistence. By a first-order Taylor expansion of function $\Psi(s) = (e^{2s}-1)/(e^{2s}+1)$ around $s = 0$, the conditional expected positional persistence is such that:

$$E_t(\rho_{i,t+1}) \simeq \Psi(0) + \Psi'(0)[\beta_i + \gamma_i E(F_{p,t+1}|F_t)] = \beta_i + \gamma_i E(F_{p,t+1}|F_t). \tag{A.18}$$

By combining (A.15)-(A.18), approximation (6.1) follows.

Appendix A.5: Parametrization of the Variance-Gamma distribution

The Variance-Gamma (VG) is a parametric family of distributions yielding a flexible yet tractable specification of third and fourth order moments. The VG distribution was first used in Finance by Madan and Seneta (1990) to describe the historical distribution of security returns. In our paper, we use the VG family to model the theoretical CS distribution of CRSP stock returns. The theoretical CS p.d.f. $h_t^*(y) = \partial H_t^*(y)/\partial y$ at month t is given by [see Seneta (2004), p. 180]:

$$h_t^*(y) = \frac{2\exp\{\gamma_t(y-c_t)/\omega_t\}}{\sqrt{2\pi\omega_t}(1/\lambda_t)^{\lambda_t}\Gamma(\lambda_t)} \left(\frac{|y-c_t|}{\sqrt{2\omega_t\lambda_t+\gamma_t^2}}\right)^{\lambda_t-1/2} K_{\lambda_t-1/2}\left(\frac{|y-c_t|\sqrt{2\omega_t\lambda_t+\gamma_t^2}}{\omega_t}\right), \tag{A.19}$$

where $K_\lambda(\cdot)$ denotes the Bessel function of the third kind [13] with index λ, $\Gamma(\cdot)$ is the Gamma function[14] and $c_t \in \mathbb{R}$, $\omega_t > 0$, $\gamma_t \in \mathbb{R}$ and $\lambda_t > 0$ are the parameters for month t. The four VG parameters c_t, ω_t, γ_t, λ_t are time-varying and stochastic. They correspond to transformations of the elements of a four-dimensional common stochastic factor $F_{d,t}$ that drives the pattern of the theoretical CS distribution of stock returns. The VG distribution in equation (A.19) is the distribution of returns $y_{i,t}$ at month t, for i varying, conditional on the observed factor $F_{d,t}$. Since the VG family of distributions can be parameterized in several alternative ways, vector $F_{d,t}$ is defined up to a one-to-one transformation. We select this transformation such that the parameters, i.e. the components of vector $F_{d,t}$, admit simple interpretations and vary without constraints in the domain $\mathbb{R}^4$. The latter condition eases the specification of a dynamic model for process (F_t) in Section 5.2. To define the parameter transformation, let us consider the first four standardized cross-sectional power moments at month t. They are given by [see Seneta (2004)]:

$$\begin{aligned}
\mu_t &= E[y_{i,t}|F_{d,t}] = c_t + \gamma_t, && \text{(A.20)}\\
\sigma_t^2 &= V[y_{i,t}|F_{d,t}] = \gamma_t^2/\lambda_t + \omega_t, && \text{(A.21)}\\
s_t &= \frac{E[(y_{i,t}-\mu_t)^3|F_{d,t}]}{\sigma_t^3} = \frac{\gamma_t}{\lambda_t}\frac{2\gamma_t^2/\lambda_t + 3\omega_t}{(\gamma_t^2/\lambda_t+\omega_t)^{3/2}}, && \text{(A.22)}\\
k_t &= \frac{E[(y_{i,t}-\mu_t)^4|F_{d,t}]}{\sigma_t^4} = 3 + \frac{3}{\lambda_t}\frac{\omega_t^2 + 4\omega_t\gamma_t^2/\lambda_t + 2\gamma_t^4/\lambda_t^2}{(\gamma_t^2/\lambda_t+\omega_t)^2}. && \text{(A.23)}
\end{aligned}$$

We have the following result, which is proved at the end of this Appendix.

Lemma A.1. *i) In the VG family, the kurtosis k_t is lower bounded, with the lower bound depending on the skewness s_t:*

$$k_t > 3(1 + s_t^2/2). \tag{A.24}$$

[13] The Bessel function of the third kind with index λ is defined as $K_\lambda(x) = \frac{1}{2}\int_0^{+\infty} t^{\lambda-1}e^{-\frac{1}{2}x(t+t^{-1})}dt$, for $x > 0$.

[14] The Gamma function is defined as $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$, for $x > 0$.

ii) Define:

$$k_t^* = k_t - 3(1 + s_t^2/2). \tag{A.25}$$

Then, the parameters $\mu_t \in \mathbb{R}$, $\sigma_t > 0$, $s_t \in \mathbb{R}$ and $k_t^* > 0$ vary independently on their domains, and are jointly in a one-to-one relationship with the original parameters $c_t \in \mathbb{R}$, $\omega_t > 0$, $\gamma_t \in \mathbb{R}$ and $\lambda_t > 0$.

The inequality (A.24) on the third and fourth order moments of the VG distribution is more restrictive than the condition valid for any distribution, namely $k_t > 1 + s_t^2$ [see Pearson (1916)]. Moreover, in the VG model the kurtosis is larger than 3, that is, the kurtosis of a Gaussian distribution. A Gaussian distribution is the limit of the VG distribution when $s_t = 0$ and $k_t^* \to 0$ [see Seneta (2004)]. Lemma A.1 *i)* suggests to consider k_t^* defined in (A.25) as a measure of excess kurtosis. Then, we define the factor $F_{d,t}$ as in equation (5.1), namely $F_{d,t} = (\mu_t, \log \sigma_t, s_t, \log k_t^*)'$. Its components are in one-to-one relationship with the parameters of the VG family from Lemma A.1 *ii)*, and they are free to vary in the unbounded domain $\mathbb{R}^4$.

Proof of Lemma A.1: We omit the time index of the parameters as it is not relevant here. Define the parameter transformations:

$$\xi = \frac{\gamma/\sqrt{\lambda}}{\sqrt{\gamma^2/\lambda + \omega}}, \qquad \eta = \frac{1}{\sqrt{\lambda}}. \tag{A.26}$$

The parameters $\mu \in \mathbb{R}$, $\sigma > 0$, $\xi \in (-1, 1)$ and $\eta > 0$ vary independently on their domains, and are in a one-to-one relationship with the original parameters $c \in \mathbb{R}$, $\omega > 0$, $\gamma \in \mathbb{R}$ and $\lambda > 0$. Indeed, the original parameters can be written as $c = \mu - \xi\sigma/\eta$, $\omega = \sigma^2(1 - \xi^2)$, $\gamma = \xi\sigma/\eta$ and $\lambda = 1/\eta^2$. Moreover, the skewness and kurtosis can be written as:

$$s = \eta\xi(3 - \xi^2), \tag{A.27}$$

$$k = 3 + 3\eta^2(1 + 2\xi^2 - \xi^4), \tag{A.28}$$

and are functions of parameters η and ξ only. From equation (A.27), when $\xi \neq 0$ we have $\eta = s/[\xi(3 - \xi^2)]$. By replacing this expression of η into equation (A.28) we get:

$$k = 3 + 3s^2 g(\xi^2), \tag{A.29}$$

where function g is defined by $g(z) = \dfrac{1 + 2z - z^2}{z(3 - z)^2}$, for $z > 0$. The function g is monotonic decreasing on the interval $(0,1)$, with $g(z) \to \infty$ as $z \to 0$ and $g(1) = 1/2$. We deduce inequality (A.24).

Defining k^* as $k^* = k - 3(1 + s^2/2)$, the parameters $\mu \in \mathbb{R}$, $\sigma > 0$, $s \in \mathbb{R}$ and $k^* > 0$ are in a one-to-one relationship with the original parameters. Indeed, given the values of $s \in \mathbb{R}$ and $k^* > 0$, we can determine uniquely the values of $\eta > 0$ and $\xi \in (-1,1)$:

i) If $s = 0$, from equations (A.27), (A.28) and the definition of k^* it follows $\xi = 0$ and $\eta = \sqrt{k^*/3}$.

ii) If $s \neq 0$, we can use equations (A.29), the definition of k^*, and the monotonicity of function g to get $\xi^2 = g^{-1}[k^*/(3s^2) + 1/2] \in (0,1)$. From equation (A.27), the sign of ξ is the same of that of s. Then, $\eta = s/[\xi(3 - \xi^2)]$. QED.

Appendix A.6: Numerical implementation of efficient positional strategies

In this appendix we provide a feasible numerical algorithm for the computation of the optimal positional portfolio allocation defined in equation (2.4). The algorithm consists in the application of the Newton-Raphson method for the solution of a maximization problem with equality constraints [see, e.g. Boyd and Vandenberghe (2004)]. Then, the conditional expectations in the gradient and the Hessian of the criterion are computed using the estimated joint model for Gaussian ranks, cross-sectional distribution and macro-factor dynamics in Sections 3 and 5.

i) Newton-Raphson algorithm with equality constraints

The maximization problem associated with equation (2.4) can be written as:

$$\begin{aligned} &\max_{\alpha} V_t(\alpha) \\ &s.t. \;\; \alpha' e = 1 \end{aligned}$$

where $\alpha = (\alpha_1, ..., \alpha_n)'$, e is an n-vector of ones, and

$$V_t(\alpha) = E_t\left[\mathscr{U}\left(H_{t+1}(\alpha' y_{t+1})\right)\right] = E_t\left[\mathscr{U}\left(H_{t+1}\left(\sum_{i=1}^{n} \alpha_i H_{t+1}^{-1}(u_{i,t+1})\right)\right)\right].$$

The associated Lagrangian function for the constrained maximization problem is:

$$\mathcal{L}_t(\alpha, \lambda) = V_t(\alpha) + \lambda(\alpha' e - 1),$$

and the first-order conditions are:

$$\begin{cases} \dfrac{\partial V_t(\alpha)}{\partial \alpha} + \lambda e = 0 \\ \alpha' e - 1 = 0 \end{cases}.$$

By applying an extended Newton's procedure based on the Taylor's expansion of the first-order conditions with respect to $(\alpha', \lambda)'$, the solution of the problem is obtained by an iterative algorithm with $(q+1)$-th step given by [see, e.g. Boyd and Vandenberghe (2004)]:

$$\begin{bmatrix} \alpha^{(q+1)} \\ \lambda^{(q+1)} \end{bmatrix} = \begin{bmatrix} \alpha^{(q)} \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{\partial^2 V_t(\alpha^{(q)})}{\partial \alpha \partial \alpha'} & e \\ e' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial V_t(\alpha^{(q)})}{\partial \alpha} \\ \alpha^{(q)\prime} e - 1 \end{bmatrix}.$$

The initial step of the algorithm uses $\alpha^{(0)} = \dfrac{1}{n} e$, that is, the equally weighted portfolio.

ii) Formulas for the gradient vector and Hessian matrix of the expected CARA positional utility function

In the case of a CARA positional utility function with $\mathscr{U}(v; \mathscr{A}) = -\exp(-\mathscr{A} v)$, where $\mathscr{A} > 0$ is the positional risk aversion parameter, written on the Gaussian rank of the portfolio return, the expected positional utility $V_t(\alpha)$ is:

$$V_t(\alpha) = -E_t \left[\exp\left\{ -\mathscr{A} H_{t+1}\left(\alpha' y_{t+1}\right) \right\} \right].$$

The gradient vector of the expected CARA positional utility function is:

$$\frac{\partial V_t(\alpha)}{\partial \alpha} = \mathscr{A} E_t \left[\exp\left\{ -\mathscr{A} H_{t+1}\left(\alpha' y_{t+1}\right) \right\} H'_{t+1}(\alpha' y_{t+1}) y_{t+1} \right], \quad \text{(A.30)}$$

where $H'_{t+1}(y) = \dfrac{dH_{t+1}(y)}{dy}$. The Hessian matrix of the expected CARA positional utility function is:

$$\frac{\partial^2 V_t(\alpha)}{\partial \alpha \partial \alpha'} = \mathscr{A} E_t \left[\exp\left\{ -\mathscr{A} H_{t+1}\left(\alpha' y_{t+1}\right) \right\} \left(-\mathscr{A} H'_{t+1}(\alpha' y_{t+1})^2 + H''_{t+1}(\alpha' y_{t+1}) \right) y_{t+1}\, y'_{t+1} \right], \quad \text{(A.31)}$$

where $H''_{t+1}(y) = \dfrac{d^2 H_{t+1}(y)}{dy^2}$.

Let us now compute the two functions $H'_{t+1}(y)$ and $H''_{t+1}(y)$. Recall from Definition 1 *ii)* in Appendix 1 that:

$$H_t(y) = \Phi^{-1}(H_t^*(y)), \tag{A.32}$$

where H_t^* is the cross-sectional c.d.f. of the VG family at date t. Therefore, we have:

$$H'_t(y) = \frac{1}{\phi[\Phi^{-1}(H_t^*(y))]} h_t^*(y),$$

where $h_t^*(y) \equiv h^*(y|F_{d,t}) = dH_t^*(y)/dy$ is the VG p.d.f. in equation (A.19) and $\phi(\cdot) = \Phi'(\cdot)$ is the p.d.f. of the standard Gaussian distribution. This allows to compute:

$$H''_t(y) = \frac{\Phi^{-1}(H_t^*(y))}{(\phi[\Phi^{-1}(H_t^*(y))])^2}(h_t^*(y))^2 + \frac{1}{\phi[\Phi^{-1}(H_t^*(y))]}\frac{dh_t^*(y)}{dy}.$$

Finally, we can re-write equation (A.31) as:

$$\frac{\partial^2 V_t(\alpha)}{\partial\alpha\partial\alpha'} \;=\; \mathscr{A} E_t\left[\exp\left\{-\mathscr{A} H_{t+1}\left(\alpha' y_{t+1}\right)\right\} \xi_{t+1}\left(\alpha' y_{t+1}\right) y_{t+1}\, y'_{t+1}\right], \tag{A.33}$$

where:

$$\begin{aligned}
\xi_t\left(y\right) &= -\mathscr{A} H'_t(y)^2 + H''_t(y) \\
&= \frac{\Phi^{-1}(H_t^*(y)) - \mathscr{A}}{(\phi[\Phi^{-1}(H_t^*(y))])^2}(h_t^*(y))^2 + \frac{1}{\phi[\Phi^{-1}(H_t^*(y))]}\frac{dh_t^*(y)}{dy}.
\end{aligned} \tag{A.34}$$

iii) Estimation of the gradient vector and Hessian matrix of the expected CARA positional utility function

Let us finally discuss the estimation of the conditional expectations in the gradient vector and Hessian matrix of the criterion given in equations (A.30) and (A.33).

The conditioning information at date t includes the past history of assets returns $\underline{y_t}$ and systematic factors $\underline{F_t}$, and the individual effects $\delta_i = (\beta_i, \gamma_i)'$ for all assets. We use that functions $H_{t+1}(\cdot)$ and $\xi_{t+1}(\cdot)$ in equations (A.32) and (A.34) involve the future factor value F_{t+1}, the asset returns are $y_{i,t+1} = H_{t+1}^{-1}(u_{i,t+1})$, and the joint process $(F_t, u_{1,t}, ..., u_{n,t})$ is Markov conditional on the individual effects. Then, the conditional expectations in equations (A.30) and (A.31) are taken with respect to future factor value F_{t+1} and Gaussian ranks $u_{i,t+1}$, given F_t, $u_{i,t}$ and δ_i for all assets in the investment universe. These conditional expectations are computed by Monte Carlo integration by simulating future ranks and factor values according to their estimated models in Sections 3 and 5. The current values of the factor F_t and ranks $u_{i,t}$, and the individual effects δ_i in the conditioning set are replaced by their estimated values.