University of St. Gallen, PhD in Economics and Finance

Advanced Time Series Econometrics

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Homework 2 SS 2005

1. Consider the AR(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \rho X_{t-1} + \epsilon_t \quad , \quad X_0 \backsim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$$

where $|\rho| < 1$ and $\varepsilon_t \backsim IIN\left(0, \sigma^2\right)$. The function

$$m\left(X_t, X_{t-1}, \rho\right) = X_t - \rho X_{t-1}$$

implies the conditional moment restrictions

$$E\left[X_t - \rho X_{t-1} | X_{t-1}\right] = 0 \quad .$$

   (a) Define (i) a set of exactly identified marginal moment conditions to estimate ρ and (ii) a set of overidentified marginal moment conditions (different from those in point (b) below) to estimate ρ.

   (b) Consider the overidentified GMM moment restrictions:

$$E\left[\left(X_t - \rho X_{t-1}\right)\begin{pmatrix} 1 \\ X_{t-1} \end{pmatrix}\right] = 0$$

   for a moment function $g : \mathbb{R}^2 \times \Theta \to \mathbb{R}^2$ given by

$$g\left(X_t, X_{t-1}, \rho\right) = \begin{pmatrix} X_t - \rho X_{t-1} \\ \left(X_t - \rho X_{t-1}\right) X_{t-1} \end{pmatrix} \quad .$$

   Under the given assumptions, determine explicitly the optimal weighting matrix Ω_0^* for the GMM estimator induced by moment function g. In particular, show that it is a diagonal matrix.

   (c) For the moment conditions in (b) verify in detail the conditions sufficient for consistency of a GMME under mixing dependence. If needed, slightly modify you setting in order to ensure consistency of the GMME.

   (d) Propose a consistent feasible estimator of the optimal weighting matrix Ω_0^*.

   (e) For the moment conditions in (b) verify in detail the conditions sufficient for asymptotic normality of the optimal GMME under mixing dependence. If needed, slightly modify you setting in order to ensure asymptotic normality of the GMME.

   (f) Give explicitly the asymptotic variance covariance matrix of the optimal GMME and compute a 95% confidence interval for ρ_0 based on a sample size $T = 100$.

2. Consider the AR(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \rho X_{t-1} + \epsilon_t \quad , \quad X_0 \backsim N\left(0, \frac{1}{1-\rho^2}\right)$$

where $|\rho| < 1$ and $\varepsilon_t \backsim IIN(0,1)$. The function

$$m(X_t, X_{t-1}, \rho) = \exp\left(X_t - \rho X_{t-1} - \frac{1}{2}\right) - 1$$

implies the conditional moment restrictions

$$E\left[m(X_t, X_{t-1}, \rho) \,|X_{t-1}\right] = 0 \quad .$$

(a) Define (i) a set of exactly identified marginal moment conditions to estimate ρ and (ii) a set of overidentified marginal moment restrictions (different from those in (b) below) to estimate ρ.

(b) Consider the overidentified GMM moment restrictions:

$$E\left[(\exp(\eta_t(\rho)) - 1)\begin{pmatrix} 1 \\ \exp(\eta_{t-1}(\rho)) \end{pmatrix}\right] = 0$$

where

$$\eta_t(\rho) = X_t - \rho X_{t-1} - \frac{1}{2},$$

corresponding to a moment function $g : \mathbb{R}^2 \times \Theta \to \mathbb{R}^2$ given by

$$g(X_t, X_{t-1}, \rho) = \begin{pmatrix} \exp(\eta_t(\rho)) - 1 \\ (\exp(\eta_t(\rho)) - 1)\exp(\eta_{t-1}(\rho)) \end{pmatrix} \quad .$$

Under the given assumptions, determine explicitly the optimal weighting matrix Ω_0^* for the GMM estimator induced by moment function g. In particular, show that it is a diagonal matrix.

(c) For the moment conditions in (b) verify in detail the conditions sufficient for consistency of a GMME under mixing dependence. If needed, slightly modify you setting in order to ensure consistency of the GMME.

(d) Propose a consistent feasible estimator of the optimal weighting matrix Ω_0^*.

(e) Compute explicity for $T \in \mathbb{N}$ the matrix

$$\frac{1}{T}\sum_{t=1}^{T} E\left(\frac{\partial g(X_t, X_{t-1}, \rho_0)}{\partial \rho}\right)$$

Is the model locally identified?

(f) Give a consistent estimator of matrix

$$J_0 = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E\left(\frac{\partial g(X_t, X_{t-1}, \rho_0)}{\partial \rho}\right)$$

(g) Which is the asymptotic covariance matrix of your optimal GMME?

3. Consider the ARCH(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$\begin{aligned} X_t &= \sqrt{h_t}\epsilon_t \quad , \\ h_t &= \omega + \alpha X_{t-1}^2 \end{aligned}$$

where $\omega > 0$, $0 < \alpha < 1$ and $\varepsilon_t \backsim IIN(0,1)$. The function

$$m(X_t, X_{t-1}, \theta) = X_t^2 / h_t(\theta) - 1,$$

where

$$h_t(\theta) = \omega + \alpha X_{t-1}^2$$

implies the conditional moment conditions

$$E\left[X_t^2/h_t(\theta) - 1 | X_{t-1}\right] = 0 \quad .$$

(a) Is $\mathcal{X}$ a geometrically mixing process (motivate your answer)?
HINT: Use the results discussed in the lecture notes.

(b) Define (i) a set of exactly identified marginal moment conditions and (ii) a set of overidentified marginal moment conditions to estimate $\theta = (\omega, \alpha)'$.

(c) Consider the GMM moment conditions:

$$E\left[\begin{pmatrix} X_t^2/h_t(\theta) - 1 \\ \left(X_t^2/h_t(\theta) - 1\right) X_{t-1}^2/h_{t-1}(\theta) \end{pmatrix}\right] = 0$$

for moment function $g : \mathbb{R}^2 \times \Theta \to \mathbb{R}^2$ given by

$$g(X_t, X_{t-1}, \theta) = \begin{pmatrix} X_t^2/h_t(\theta) - 1 \\ \left(X_t^2/h_t(\theta) - 1\right) X_{t-1}^2/h_{t-1}(\theta) \end{pmatrix} \quad .$$

Discuss the sufficient conditions for consistency of a GMME under mixing dependence. If needed, slightly modify you setting in order to ensure consistency of the GMME.
HINT: You might try to consult the paper Carrasco and Chen (2005), in order to discuss existence of relevant moments of $m(X_t, X_{t-1}, \theta)$.

(d) Compute

$$V_0 = \lim_{T\to\infty} Var\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} g(X_t, X_{t-1}, \theta_0)\right)$$

and propose a consistent feasible estimator of it.

(e) Assuming the conditions for asymptotic normality of your GMME to be satisfied, explain how you would compute a 95% confidence interval for parameter α_0.

4. For a given kernel of class $\mathcal{K}_1$ and a bandwidth γ_T such that $\gamma_T \to \infty$ and $\gamma_T = o(T)$, consider the spectral density covariance matrix estimator (see the lecture notes)

$$\widehat{V}_T = \sum_{j=-(T-1)}^{T-1} k(j/\gamma_T)\,\widehat{\Gamma}_T(j) \quad .$$

   (a) Verify that the truncated kernel is not of class $\mathcal{K}_2$.

   (b) Verify that the Parzen kernel is of class $\mathcal{K}_2$.

5. For the AR(1) setting of Exercise 1 set $\sigma^2 = 1$ and consider the scalar GMM moment function:

$$g(X_t, \rho) = X_t^2 - \frac{1}{1-\rho^2},$$

defining the GMM moment restriction

$$E\left(X_t^2 - \frac{1}{1-\rho^2}\right) = 0 \quad .$$

   (a) i. For given $s \geq t$ and every $|\rho| < 1$ compute

$$Cov\left(X_s^2, X_t^2\right) = Cov\left(g(X_s, \rho), g(X_t, \rho)\right) \quad .$$

   ii. Define

$$V_0 = \lim_{T\to\infty} \frac{1}{T} Var\left(\sum_{t=1}^{T} g(X_t, \rho_0)\right)$$

   and give an explicit expression of V_0 using the findings in (i).

   iii. Give two consistent positive definite estimators of

$$V_0 = \lim_{T\to\infty} \frac{1}{T} Var\left(\sum_{t=1}^{T} g(X_t, \rho_0)\right)$$

   and check the sufficient conditions for consistency in the case where ρ_0 is known. At which rate is the bandwidth γ_T allowed to grow with sample size? What happens to your results if $\rho = 0$?

   iv. Give two consistent positive definite estimators of V_0 and check the sufficient conditions for consistency in the case where ρ_0 is unknown. At which rate is the bandwidth γ_T allowed to grow with sample size? What happens to your results if $\rho = 0$?

6. Let the MA(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$ be defined by

$$X_t = \rho\epsilon_{t-1} + \epsilon_t \quad , \quad t \in \mathbb{N}$$

where $|\rho| < 1$ and $(\varepsilon_t)_{t\in\mathbb{Z}} \backsim IID(0,1)$ is Bernoully distributed and such that $P(\varepsilon_t = 1) = P(\varepsilon_t = -1) = 1/2$. Consider the scalar GMM moment function:

$$g(X_{t-1}, X_t, \rho) = X_t X_{t-1} - \rho,$$

defining the GMM moment restriction

$$E(X_{t-1}X_t - \rho) = 0 \quad .$$

(a) Show that

$$\begin{aligned} V_0 &= \lim_{T\to\infty} \frac{1}{T} Var\left(\sum_{t=1}^{T} g(X_{t-1}, X_t, \rho_0)\right) \\ &= E\left[g^2(X_{t-1}, X_t, \rho_0)\right] + 2E\left[g(X_{t-1}, X_t, \rho_0)\, g(X_t, X_{t+1}, \rho_0)\right] \end{aligned}$$

and give an explicit expression of V_0.

(b) Consider the simple "finite lag" variance estimator

$$\widehat{V}_T = \frac{1}{T}\sum_{t=1}^{T} g^2(X_{t-1}, X_t, \rho_0) + \frac{2}{T}\sum_{t=1}^{T-1} g(X_{t-1}, X_t, \rho_0)\, g(X_t, X_{t+1}, \rho_0)$$

where, for simplicity, $\rho_0 = 0$. Fix $T = 3$ and show that

$$\widehat{V}_3 = \frac{1}{3}\left[(X_1X_0 + X_2X_1)^2 + X_3X_2^2(X_3 + 2X_1)\right] \quad .$$

(c) Show that

$$P\left(\widehat{V}_3 < 0\right) > 0 \quad .$$

Discuss and interpret your findings! How can you obtain an estimator $\widehat{V}_T$ such that $P\left(\widehat{V}_T < 0\right) = 0$?

HINT: Compute

$$P\left(\{X_0 = -X_2\} \cap \left\{X_3^2 + 2X_1X_3 < 0\right\}\right) \quad .$$