University of St. Gallen, PhD in Economics and Finance

Advanced Time Series Econometrics

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Homework 1 SS 2005

1. Consider the Gaussian MA(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \rho\epsilon_{t-1} + \epsilon_t \quad , \quad t \in \mathbb{N}$$

where $|\rho| < 1$ and $(\varepsilon_t)_{t\in\mathbb{Z}} \backsim IIN\left(0, \sigma^2\right)$.

   (a) Argue that $\{X_t : t \in \mathbb{N}\}$ is strictly stationary.

   (b) Show that $\mathcal{X}$ is an L_1-mixingale and determine the corresponding mixingale constants.

   (c) Show that $\mathcal{X}$ is an $\alpha-$mixing process of size $\phi_0 = -\infty$.

   (d) Show that $\mathcal{X}$ is an asymptotically stationary process by testing explicitly the weak convergence of

$$\mu^T := \frac{1}{T}\sum_{t=1}^{T}\mu_{X_t}$$

   to a suitable limit probability measure μ.

2. Consider the Gaussian AR(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \rho X_{t-1} + \epsilon_t \quad , \quad X_0 = 0$$

where $|\rho| < 1$ and $\varepsilon_t \backsim IIN\left(0, \sigma^2\right)$.

   (a) Determine explicitly for every $t \in \mathbb{N}$ the finite dimensional distribution of X_t.

   (b) Show that

$$\mu_{X_t} \Rightarrow N\left(0, \frac{\sigma^2}{1-\rho^2}\right) \tag{1}$$

   HINT: Use the following characterization of weak convergence (see also Davidson (1994), Thm 22.17):

$$\mu_{X_t} \Rightarrow N\left(0, \frac{\sigma^2}{1-\rho^2}\right) \Leftrightarrow \phi_t(u) \to \phi(u) \quad ,$$

   where ϕ_t is the characteristic function of μ_{X_t} and ϕ the characteristic function of the $N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$ distribution [See Davidson (1994), Chapter 11, for a review of characteristic functions].

(c) Show that $\mathcal{X}$ is an asymptotically stationary process by testing explicitly the weak convergence of

$$\mu^T := \frac{1}{T}\sum_{t=1}^{T}\mu_{X_t}$$

to a suitable limit probability measure μ. Give μ explicitly.

(d) Show, with a simple counterexample, that $\mathcal{X}$ is not strictly stationary.

(e) Show that $\mathcal{X}$ is an L_1-mixingale and determine the corresponding mixingale size and constants.

3. Consider the random walk process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = X_{t-1} + \epsilon_t \quad , \quad X_0 = \mu \in \mathbb{R}$$

where $\varepsilon_t \backsim IID\left(0, \sigma^2\right)$.

(a) Show that $\mathcal{X}$ cannot be a mixing process.
HINT: Consider the autocovariance function of $\mathcal{X}$.

(b) Show that if $\mu \neq 0$ process $\mathcal{X}$ is not a mixingale.

(c) Show that if $\mu = 0$ process $\mathcal{X}$ is an L_1-mixingale with constants $c_t = t^2$ and a null sequence (ζ_m) given by $\zeta_m = 1/m$.

(d) Show that if $\mu = 0$ process $\mathcal{Y} := \{Y_t : t \in \mathbb{N}\}$ defined by $Y_t = X_t/t$ is an L_1-mixingale with constants $c_t = t$ and a null sequence (ζ_m) given by $\zeta_m = 1/m$.

(e) Using the results in (c) and (d), can you apply a LLN for mixingales to show a.s. convergengence of at least one of the two following sequences:

$$\left\{\sum_{t=1}^{T} X_t : T \in \mathbb{N}\right\} \quad , \quad \left\{\sum_{t=1}^{T} \frac{X_t}{t} : T \in \mathbb{N}\right\} \quad ?$$

4. Consider a simple AR(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \frac{1}{2}X_{t-1} + \epsilon_t \quad , \quad X_0 = \epsilon_0$$

where $|\rho| < 1$ and ε_t is an IID binomial sequence such that $P(\epsilon_t = 0) = P(\epsilon_t = 1) = \frac{1}{2}$.

(a) Show that $\mathcal{X}$ is not an $\alpha-$mixing process. To this end, perform the following steps:

i. Define

$$B_t = \left\{\frac{k}{2^t} : k = 0, 2, 4, .., 2\left(2^t - 1\right)\right\}$$

and show that

$$X_h \in B_h \Leftrightarrow X_0 = 0 \quad , \quad h \in \mathbb{N} \quad .$$

ii. Compute

$$P(X_h \in B_h) \quad , \quad P(X_0 = 1) \quad , \quad P(\{X_0 = 1\} \cap \{X_h \in B_h\}),$$

to obtain a lower bound for the mixing coefficient $\alpha(h)$ and take limits as $h \to \infty$.

(b) Explain why the results obtained are not in contradiction to the examples of mixing processes discussed in the lecture notes.

5. Let $\mathcal{X}$ be a zero mean (adapted) $\alpha-$mixing process of size $-\phi_0$, which is L_r-bounded, where $r > 1$. Apply Serfling Lemma to prove that:

   (a) $p = 2$: $\mathcal{X}$ is an L_2-mixingale of size $-1/2$ if $-\phi_0 = -r/(r-2)$.

   (b) $1 < p < 2$: $\mathcal{X}$ is an L_p-mixingale of size -1 if $-\phi_0 = -rp/(r-p)$.

6. Consider the AR(1) process $\mathcal{X} := \{X_t : t \in \mathbb{N}\}$, defined by

$$X_t = \rho X_{t-1} + \epsilon_t \quad , \quad X_0 \backsim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)$$

where $|\rho| < 1$ and $\varepsilon_t \backsim IIN(0, \sigma^2)$. We know from the lecture notes that this process is geometrically mixing. The OLS estimate $\widehat{\rho}_T$ of ρ is defined by

$$\widehat{\rho}_T = \frac{\sum_{t=1}^{T-1} X_t X_{t-1}}{\sum_{t=1}^{T-1} X_t^2} = \rho + \frac{\sum_{t=1}^{T-1} X_{t-1}\epsilon_t}{\sum_{t=1}^{T-1} X_t^2} \quad .$$

Check explicitly the conditions needed to apply a suitable LLN to show

$$\frac{1}{T-1}\sum_{t=1}^{T-1} X_{t-1}\epsilon_t \overset{a.s}{\to} 0$$

$$\frac{1}{T-1}\sum_{t=1}^{T-1} X_t^2 \overset{a.s}{\to} \frac{\sigma^2}{1-\rho^2}$$

and to imply the (strong) consistency of $\widehat{\rho}_T$. To this end, perform the following steps:

   (a) Show that process $\mathcal{Y} := \{Y_t : t \in \mathbb{N}^*\}$, defined by

$$Y_t = X_{t-1}\epsilon_t$$

   is a zero mean L_p-mixingale, for some $1 < p \leq 2$, and apply a suitable LLN to $\mathcal{Y}$.

   (b) Show that process $\mathcal{Y} := \{Y_t : t \in \mathbb{N}^*\}$, defined by

$$Y_t = X_t^2 - E\left(X_t^2\right)$$

   is a zero mean L_p-mixingale, for some $1 < p \leq 2$, and apply a suitable LLN to $\mathcal{Y}$.

(c) Show that

$$E\left(X_t^2\right) \to \frac{\sigma^2}{1-\rho^2} > 0 \quad .$$

7. In the setting of Exercise 6 show that, under the given conditions

$$\sqrt{T}\left(\widehat{\rho}_T - \rho\right) \xrightarrow{D} N\left(0, 1-\rho^2\right)$$

Perform the following steps:

(a) Argue that process $\mathcal{Y} := \{Y_t : t \in \mathbb{N}^*\}$ defined by

$$Y_t = X_{t-1}\epsilon_t$$

is geometrically strong mixing.

(b) Check the conditions to apply Herrndorf's CLT for mixing processes.

(c) Apply Herrndorf's CLT for mixing processes.